

THERMAL STRESSED STATES OF THE BODIES OF REVOLUTION MADE OF FUNCTIONALLY GRADED MATERIALS

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We propose a method for the determination of the thermal stressed state of elastic bodies of revolution made of functionally graded materials. The formulation of the mathematical problem contains a thermal loading. A numerical algorithm for the solution of the corresponding problem is developed and realized for the problems of heat conduction and thermoelasticity. The thermoelastic state of a hollow cylinder is computed for a given mode of heating and different heat-conduction coefficients, elasticity moduli, specific heat capacities, and linear coefficients of thermal expansion. These results enable us to plot the graphic dependences of the thermal stressed state of a hollow cylinder.

Keywords: thermal stressed state, body of revolution, functionally graded materials, finite-element method, hollow cylinder.

Introduction

In recent years, we observe rapid progress in the development of nanotechnologies. Thus, it becomes possible to produce materials with given physicochemical characteristics, which are continuous functions of the space coordinate and temperature. These materials are called functionally graded. The structures made of elastic functionally graded materials are frequently heated and cooled. Under the action of temperature fields, we observe the formation of temperature stresses in these materials. These stresses may exceed admissible values and lead to the failures of structures. The main part of available investigations deals with bodies of revolution. The analysis of thermal stressed state formed in the bodies of revolution made of functionally graded materials subjected to thermal loading was carried out in numerous works and, in particular, in [1–4]. These works also mainly deal with the bodies with canonical shapes of revolution. As compared with the results of investigations in this field available in the literature, we propose a new mathematical procedure aimed at determination of the thermal stressed state of bodies of revolution made of elastic functionally graded materials whose shapes are different from canonical ones. The thermophysical characteristics of the body depend on the space coordinate and temperature. The proposed method includes the mathematical formulation of the problem, the development of a numerical algorithm for finding its solution, and the software realization of the numerical algorithm.

Statement of the Problem

We assume that, under the analyzed thermal loads, the stressed state of the body does not affect its temperature. In other words, the problem of determination of the stress-strain state of the body is formulated in the quasi-static statement (in displacements).

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The body of revolution occupies a domain Ω of the Euclidean space R^3 and is bounded by a surface Γ continuous in a sense of Lipschitz. The body is referred to a curvilinear cylindrical coordinate system $Or\varphi z$. In a part Γ_u of the surface of the body Γ , we specify displacements

$$\vec{u} = \vec{u}_0,$$

whereas the other part of the surface Γ_σ is subjected to the action of force loading characterized by a vector $\vec{p}$ and, in addition, $\Gamma_u \cup \Gamma_\sigma = \Gamma$. The inhomogeneous structure of the material of the body is represented by physicommechanical characteristics regarded as functions of the space coordinate and temperature. The body is also subjected to axisymmetric technological heating by the ambient medium with a temperature $t_c(\tau)$ through a part of the surface Γ_t , by a heat flux $q(\tau)$ through a part Γ_q of the surface ($\Gamma_t \cup \Gamma_q = \Gamma$), and by distributed heat sources with intensity $Q(\vec{r}, \tau)$, ($\vec{r} \in \Omega$). The temperature field in the body is described by the well-known heat-conduction equation [5]:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(rk \frac{\partial t}{\partial r} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial t}{\partial z} \right) + Q = c\rho \frac{\partial t}{\partial \tau} \quad (1)$$

with the following initial and boundary conditions:

$$\begin{aligned} t(r, z, 0) = t_0(r, z); \quad \left[k \left(\frac{\partial t}{\partial r} n_r + \frac{\partial t}{\partial z} n_z \right) + \alpha(t - t_c) \right]_{\Gamma_t} &= 0, \\ \left[k \left(\frac{\partial t}{\partial r} n_r + \frac{\partial t}{\partial z} n_z \right) + q \right]_{\Gamma_q} &= 0. \end{aligned} \quad (2)$$

Here, $k(s, t)$ is the heat-conduction coefficient, $c(s, t)$ is the specific heat capacity, $\rho(s, t)$ is density; α is the heat-transfer coefficient, n_r and n_z are the components of the outer normal to the surface of the body, s is the space coordinate in the direction of inhomogeneity of the physicommechanical properties of the material, τ is the running time, and r and z are the space coordinates of a point.

We assume that the relationship between the components of the stress $\{\sigma\}$ and strain $\{\varepsilon\}$ tensors has the form [1]

$$\{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon_t\}), \quad (3)$$

where $[D]$ is the matrix of elastic characteristics and $\{\varepsilon_t\}$ is the tensor of temperature loading.

In a cylindrical coordinate system (for the axisymmetric case), these tensors can be represented in terms of their components as follows:

$$\begin{aligned} \{\sigma\}^T &= [\sigma_{rr} \ \sigma_{\varphi\varphi} \ \sigma_{zz} \ \sigma_{rz}], \quad \{\varepsilon\}^T = [\varepsilon_{rr} \ \varepsilon_{\varphi\varphi} \ \varepsilon_{zz} \ \varepsilon_{rz}], \\ \{\varepsilon_t\}^T &= [1 \ 1 \ 1 \ 0] \int_{t_0}^t \alpha_t(s, \varsigma) d\varsigma, \end{aligned} \quad (4)$$

where $\alpha_t(s, t)$ is the coefficient of linear thermal expansion.

The matrix of elastic characteristics takes the form

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & 0.5-\nu \end{bmatrix}, \quad (5)$$

where $E(s, t)$ is the modulus of elasticity and $\nu(s, t)$ is Poisson's ratio.

In the domain Ω , we also have the following balance equation and boundary conditions:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0, \quad \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0; \quad (6)$$

$$n_r \sigma_{rr} + n_z \sigma_{zr} - p_r = 0, \quad n_r \sigma_{rz} + n_z \sigma_{zz} - p_z = 0 \quad \text{on } \Gamma_\sigma, \quad (7)$$

$u_r = u_r^0$, and $u_z = u_z^0$ on Γ_u .

We consider the case of small deformations. The relationships between the components of the strain tensor and the vector of displacements take the form:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\varphi\varphi} = \frac{u_r}{r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}. \quad (8)$$

We pose the problem in the quasistatic statement and in displacements. Dependences (1)–(8) form the complete system of equations for the determination of the temperature field, the components of the vector of displacements, the strain and stress tensors at a given temperature of the medium t_c , the heat flux q , the external force load $\vec{p}$, and the power of internal heat sources Q .

Procedure of Solution of the Problem of the Thermal Stressed State of the Body of Revolution Made of a Functionally Graded Material

Note that the geometric configuration of the domain occupied by the body is often quite complicated and the system of differential equations is nonlinear. Therefore, for the solution of the problems, we used the method of weighed residues in combination with the finite-element approach [5, 6]. This enables us to find efficient approximate solutions of the posed problems. In the proposed approach, the algorithm of solution of a given problems includes the partition of the domain occupied by the body into elements, the approximation of the unknown functions in elements of the partition, the construction, on this basis, of a system of algebraic equations for the unknown functions at the nodes of elements, and the choice of an algorithm for the solution of this system.

According to the concept of the finite-elements method, we split the domain Ω into finitely many n_e subdomains $\Omega^{(e)}$ such that the following conditions are satisfied:

$$\forall e, m: e = 1, \dots, n_e, \quad m = 1, \dots, n_e, \quad \Omega^{(e)} \cap \Omega^{(m)} = \emptyset, \quad e \neq m, \quad \text{and} \quad \Omega = \bigcup_{e=1}^{n_e} \Omega^{(e)}.$$

In the partition of the domain, we use axisymmetric biquadratic (8 nodes) isoparametric finite elements. The approximation of the scalar quantity t (temperature) in a finite element is realized with the help of a shape function in the form [6]

$$t^{(e)} = \begin{bmatrix} N_1^{(e)} & \dots & N_8^{(e)} \end{bmatrix} \begin{Bmatrix} T_1^{(e)} \\ \vdots \\ T_8^{(e)} \end{Bmatrix}, \quad (9)$$

where $N_i^{(e)}$ is the base function of the element, which is called a shape function; moreover, $N_i^{(e)} = 1$ at the node i and is equal to zero at all other nodes, and $T_i^{(e)}$ is the value of the function t at the i th node. We represent the vector quantity $\vec{u}$ (displacement) in terms of its components regarded as unknown scalar quantities. At each node, we have two unknown quantities, which can be represented in the form [6]:

$$\begin{Bmatrix} u_r^{(e)} \\ u_z^{(e)} \end{Bmatrix} = \begin{bmatrix} N_1^{(e)} & 0 & \dots & N_8^{(e)} & 0 \\ 0 & N_1^{(e)} & \dots & 0 & N_8^{(e)} \end{bmatrix} \begin{Bmatrix} U_1^{(e)} \\ \vdots \\ U_{16}^{(e)} \end{Bmatrix}, \quad (10)$$

where $U_1^{(e)}, \dots, U_{15}^{(e)}$ and $U_2^{(e)}, \dots, U_{16}^{(e)}$ are the values of the components u_r and u_z at the nodes of the elements, respectively.

Heat-Conduction Problem. Consider the heat-conduction problem (1), (2). By using the method of weighed residues (in particular, the Galerkin method) in combination with the finite-element method and taking into account the boundary conditions (2), we reduce the heat-conduction equation (1) to the form [6]

$$[C_t] \frac{d\{T\}}{d\tau} + [K_t] \{T\} = \{f_t\}, \quad (11)$$

where $[C_t]$, $[K_t]$, and $\{f_t\}$ are the corresponding temperature matrices of stiffness and the loading vector, which take the form:

$$\begin{aligned} [C_t] &= \sum_{j=1}^{n_e} \int_{\Omega^{(j)}} \rho c [N^{(j)}]^T [N^{(j)}] d\Omega, \\ [K_t] &= \sum_{j=1}^{n_e} \int_{\Omega^{(j)}} [B^{(j)}]^T [D_t^{(j)}] [B^{(j)}] d\Omega + \sum_{j=1}^{n_{e1}} \int_{\Gamma_t^{(j)}} \alpha [N^{(j)}]^T [N^{(j)}] d\Gamma, \\ \{f_t\} &= \sum_{j=1}^{n_e} [N^{(j)}]^T Q d\Omega + \sum_{j=1}^{n_{e1}} \int_{\Gamma_t^{(j)}} \alpha [N^{(j)}]^T t_c d\Gamma + \sum_{j=1}^{n_{e2}} [N^{(j)}]^T q d\Gamma, \\ [B^{(j)}] &= \begin{bmatrix} \frac{\partial N_1^{(j)}}{\partial r} & \dots & \frac{\partial N_8^{(j)}}{\partial r} \\ \frac{\partial N_1^{(j)}}{\partial z} & \dots & \frac{\partial N_8^{(j)}}{\partial z} \end{bmatrix}, \quad [D_t^{(j)}] = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}. \end{aligned}$$

We now apply the method of finite differences (we split the segment $[0, \tau_k]$ by a discrete collection of points $\tau_0, \tau_1, \dots$ with steps $\Delta\tau$). Then Eq. (11) yields the following system of nonlinear algebraic equations:

$$([C_t] + \theta \Delta\tau [K_t]) \{T\}_{n+1} - ([C_t] - (1 - \theta) \Delta\tau [K_t]) \{T\}_n = \{f_t\}_n, \quad (12)$$

where the subscripts n and $n+1$ denote the values of a quantity at the times τ_n and τ_{n+1} , respectively.

Choosing the collocation points θ , we transform relations (12) into one of well-known finite-difference schemes for Eq. (11), namely (see [6]), into Euler's scheme for $\theta = 0$, or into the Crank–Nicolson scheme for $\theta = 0.5$, or into the Galerkin method for $\theta = 0.66667$. The nonlinear system of algebraic equations (12) can be solved by the iterative method [7].

Problem of Thermoelasticity. By using the method of weighed residues in combination with the method of finite elements, we can reduce the systems of equations (3)–(8) used for the description of the behavior of a thermoelastic body to the form

$$[K_u] \{U(\tau)\} = \{F_0(\tau)\}, \quad (13)$$

where $[K_u]$ and $\{F_0\}$ are, respectively, the stiffness matrix and the vector of loading, which take the form

$$\begin{aligned} [K_u] &= \sum_{e=1}^{n_e} \int_{\Omega^{(e)}} [B_u^{(e)}]^T [D_0^{(e)}] [B_u^{(e)}] d\Omega; \\ \{F_0\} &= \sum_{e=1}^{n_e} \int_{\Omega^{(e)}} [B_u^{(e)}]^T [D_0^{(e)}] \{\epsilon_t^{(e)}\} d\Omega \\ &\quad + \sum_{e=1}^{n_e} \int_{\Omega^{(e)}} [N_u^{(e)}]^T \{X^{(e)}\} d\Omega + \sum_{e=1}^{n_{e1}} \int_{\Gamma_\sigma^{(e)}} [N_u^{(e)}]^T \{p^{(e)}\} d\Gamma; \\ [B_u^{(e)}] &= [Z] [N_u^{(e)}] = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} N_1 & 0 & \dots & N_8 & 0 \\ 0 & N_1 & \dots & 0 & N_8 \end{bmatrix}. \end{aligned}$$

The nonlinear system of algebraic equations (13) is also solved by the iterative method.

Thermal Stressed State of a Hollow Cylinder under the Conditions of Heating and Cooling

The proposed model is used for the determination of the thermal stressed state of a hollow cylinder (Fig. 1) made of an elastic functionally graded (in the direction of the Or -axis) material with a height $z_1 = 0.1$ m.

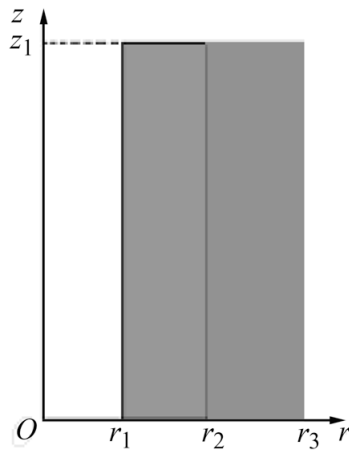


Fig. 1. Axial section of the cylinder.

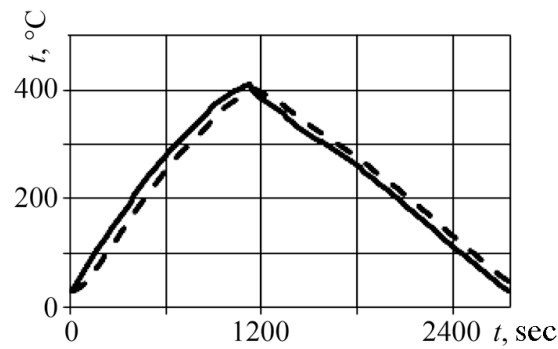


Fig. 2. Temperature mode of heating of the hollow cylinder (the solid and dashed lines correspond to the outer and inner surfaces, respectively).

Table 1
Dependences of the Heat-Conduction Coefficient k
on the Coordinate r and Temperature t .

k , W/(m·K)	r , m			
	0.206	0.211	0.216	
t , °C	0	1.50	0.60	1.05
	80	1.55	0.65	1.11
	160	1.60	0.68	1.14
	240	1.61	0.70	1.16
	320	1.62	0.72	1.17
	400	1.63	0.78	1.21
	480	1.66	0.79	1.23

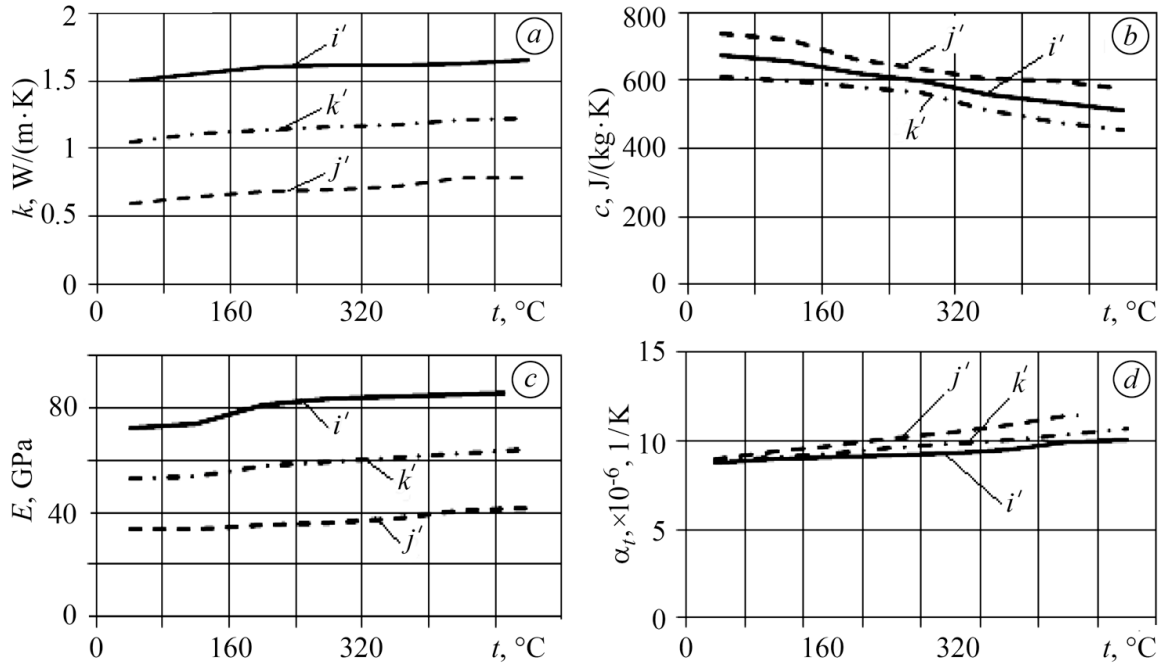


Fig. 3. Thermal characteristics of the functionally graded material of a hollow cylinder: (a) heat-conduction coefficient, (b) specific heat capacity, (c) elasticity modulus, (d) coefficient of linear thermal expansion (i' , j' , and k' are the points of measurements).

In Fig. 2, we present the temperature mode of heating of a hollow cylinder typical of various technological processes.

In Fig. 3, we present the thermal characteristics of a functionally graded material typical of silicate materials (the measurements were carried out at points with the coordinates $r_1 = 0.206$ m, $r_2 = 0.211$ m, and $r_3 = 0.216$ m).

Since the physicomaterial characteristics of material of the cylinder are functions both of temperature and of the space coordinate along the Or -axis, their discretization is performed according to the scheme illustrated by an example of heat-conduction coefficient k . First, we represent the plot of heat-conduction coefficient as a function of temperature and the coordinate r (Fig. 3a) in the form of Table 1.

At any temperature t and for any coordinate r , the heat-conduction coefficient k can be found according to the following algorithm (linear approximation):

$$k(t, r) = \begin{bmatrix} N_{ij} & N_{i+1j} & N_{ij+1} & N_{i+1j+1} \end{bmatrix} \begin{Bmatrix} k_{ij} \\ k_{i+1j} \\ k_{ij+1} \\ k_{i+1j+1} \end{Bmatrix} = [N] \{K\}, \quad (14)$$

where k_{ij} , k_{i+1j} , k_{ij+1} , and k_{i+1j+1} are the known tabular values of the heat-conduction coefficient bounding the unknown value of k for any known temperature t and coordinate r and N_{ij} , N_{i+1j} , N_{ij+1} , and N_{i+1j+1} are shape functions of the form

$$N_{ij} = \left(1 - \frac{r - r_i}{r_{i+1} - r_i} \right) \left(1 - \frac{t - t_j}{t_{j+1} - t_j} \right),$$

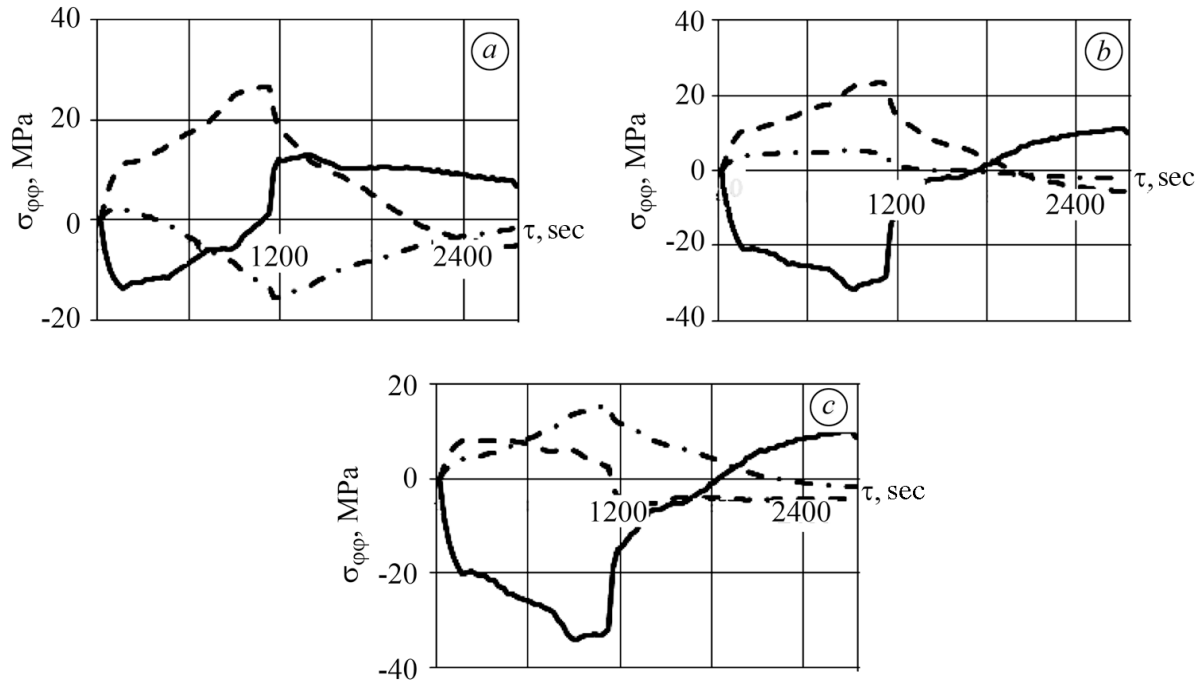


Fig. 4. Thermal stressed state (the solid, dashed, and dot-dashed lines correspond to the outer, inner, and median surfaces, respectively).

$$N_{i+1j} = \left(\frac{r - r_i}{r_{i+1} - r_i} \right) \left(1 - \frac{t - t_j}{t_{j+1} - t_j} \right),$$

$$N_{ij+1} = \left(1 - \frac{r - r_i}{r_{i+1} - r_i} \right) \left(\frac{t - t_j}{t_{j+1} - t_j} \right),$$

$$N_{i+1j+1} = \left(\frac{r - r_i}{r_{i+1} - r_i} \right) \left(\frac{t - t_j}{t_{j+1} - t_j} \right).$$

In a similar way, we approximate the other physico-mechanical characteristics of the material.

The thermal stressed state of a hollow cylinder is determined by using algorithms (12) and (13). For the proposed mode of heating (see Fig. 2), we compute the thermal stressed state of the cylinder in the following three cases: The first case (Fig. 4a) corresponds to the thermal characteristics of the material at the measurement points i' , j' , and k' (Fig. 3), the second case (Fig. 4b) corresponds to the points i' , k' , and j' , and the third case (Fig. 4c) corresponds to k' , i' , and j' .

It was discovered (Fig. 4) that the level of tensile stresses noticeably depends on the properties of the functionally graded material in the bulk of the cylinder. The maximum level of tensile stresses equal to 25 MPa corresponds to the characteristics of the material at the measurement points i' , j' , k' , while the minimum level equal to 18 MPa corresponds to k' , i' , and j' . This enables one to model the strength of the structure in the course of operation by the proper choice of the inhomogeneous physico-mechanical characteristics of the material. The models of the thermal stressed state of functionally graded materials are applicable to the investigation of fracture processes with regard for the hydrogenation and dehydrogenation of materials [8–10].

CONCLUSIONS

We propose a numerical method for evaluation of stresses in bodies of revolution made of elastic functionally graded materials subjected to heating and cooling. This method enables us to solve a broad class of problems related to the investigation of parameters characterizing the thermal and mechanical processes running in structures. The numerical results (Fig. 4) demonstrate that the actual strength of structures and their service life strongly depend on the choice of the inhomogeneous physicomaterial characteristics of the material. The developed numerical-analytic model can be improved by using the results obtained in [8–10].

REFERENCES

1. S. M. Aizikovich, V. M. Aleksandrov, A. S. Vasil'ev, L. I. Krenev, and I. S. Trubchik, *Analytic Solution of Mixed Axisymmetric Problems for Functionally Graded Media* [in Russian], Fizmatlit, Moscow (2011).
2. V. S. Nikishin and G. S. Shapiro, *Problems of Elasticity Theory for Multilayer Media* [in Russian], Nauka, Moscow (1973).
3. J. Aboudi, M. J. Pindera, and S. M. Arnold, "Higher-order theory for functionally graded materials," *Composites*, **30**, No. 8, 777–832 (1999).
4. A. Kawasaki and R. Watanabe, "Thermal fracture behavior of metal/ceramic functionally graded materials," *Eng. Fract. Mech.*, No. 69, 1713–1728 (2002).
5. E. M. Irza, "Mathematical model of the formation of residual stresses in glass bodies of revolution in the process of cooling," *Fiz.-Khim. Mekh. Mater.*, **44**, No. 2, 14–18 (2008); **English translation: Mater. Sci.**, **44**, No. 2, 156–162 (2008).
6. O. C. Zienkiewicz and R. L. Taylor, *Finite Element Method: The Basis*, Vol. 1, Butterworth Heinemann, London (2000).
7. L. I. Turchak, *Foundations of Numerical Methods* [in Russian], Nauka, Moscow (1987).
8. L. M. Ivas'kevych, "Influence of temperature and cyclic loading on hydrogen embrittlement of nickel refractory alloys," *Fiz.-Khim. Mekh. Mater.*, **47**, No. 1, 70–75 (2011); **English translation: Mater. Sci.**, **47**, No. 1, 76–81 (2011).
9. M. H. Stashchuk, "Influence of hydrogen concentration on the stresses in a solid metallic cylinder," *Fiz.-Khim. Mekh. Mater.*, **53**, No. 6, 73–79 (2017); **English translation: Mater. Sci.**, **53**, No. 6, 823–831 (2017).
10. O. E. Andreikiv, V. R. Skal's'kyi, and O. V. Hembara, "A method for the investigation of high-temperature hydrogen-assisted fracture of bimetallic structural elements," *Fiz.-Khim. Mekh. Mater.*, **36**, No. 4, 15–22 (2000); **English translation: Mater. Sci.**, **36**, No. 4, 489–498 (2000).