

A VARIATIONAL-RECURSIVE METHOD OF GENERALIZED SEPARATION OF VARIABLES FOR SOLVING MULTIDIMENSIONAL INTEGRAL EQUATIONS

M. M. Voitovich and S. A. Yaroshko

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We describe a recursive method of solving multidimensional integral equations. An approximate separation of variables holds at each step from the condition for minimizing a suitable functional. The original problem is reduced to a sequence of one-dimensional problems. Several versions of the algorithm are given.

Introduction. In solving multidimensional applied problems separation of variables makes it possible, first, to reduce the dimension of the problem, and second, to present the solution compactly. When it is not possible to achieve this exactly, one can separate the variables approximately, using the condition for minimizing a certain functional.

Problems involving approximate separation of variables arise in many contexts: in approximating operators and functions of several variables and in the discrete analogs of such problems [1–3, 8–10, 14, 17]; in solving multidimensional boundary-value problems [6, 7, 13, 15], integral equations in multidimensional domains [4, 5, 18], and inverse problems, including nonlinear problems [11, 12, 18]. Papers devoted to this topic can be provisionally divided into several groups. The papers [1–3, 8–10, 14, 17] study problems of approximate replacement of functions or operators by sums of terms with separated variables. In [7, 13] a method of seeking approximate solutions of boundary-value problems in the form of a single function with variables separated was proposed. The two approaches were combined in [4–6, 11, 12, 15, 18]. All of these papers were based on variational formulations of the problems.

The problem of the representability of solutions of differential and operator-differential equations in the form of a finite number of terms with variables separated has been studied in the generalized method of separation of variables [16], as a generalization of the Fourier method.

In the present paper we develop a method of generalized separation of variables proposed in [4, 5, 11], adapted for multidimensional integral and matrix equations and their variational analogs. Such an approach is used in [6, 15] to solve boundary-value problems.

The idea of the method is to represent the solution as a sum of terms with variables separated, which can be computed successively from the condition for minimizing corresponding functionals. The problem reduces to a sequence of one-dimensional problems. Several versions of the method are given, based on different variational formulations.

The general outline of the algorithm. We consider an n -dimensional linear inhomogeneous Fredholm integral equation of second kind

$$Au \equiv \int_D K(x, x')u(x) dx - \lambda u(x') = f(x'), \quad (1)$$

where D is a bounded domain in R^n , $x = (x_1, \dots, x_n)$, $x' = (x'_1, \dots, x'_n)$, $u \in L^2(D)$, $f \in L^2(D)$, $K \in L^2(D \times D)$, and λ does not belong to the spectral kernels of K .

Let a_j and b_j be the j th coordinates of the faces of the n -dimensional parallelepiped circumscribed about D , $a_j \leq x_j \leq b_j$, and D_j the intersection of the domain D with the hyperplane $x_j = \text{const}$. Then for any function $\mathcal{F}(x)$ that is integrable in D and any j , $1 \leq j \leq n$, we have the equality

$$\int_D \mathcal{F}(x)(x) dx = \int_{a_j}^{b_j} dx_j \int_{D_j} \mathcal{F}(x) d\hat{x}_j, \quad (2)$$

where $d\hat{x}_j$ is the element of volume in the domain D_j .

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We seek a solution of Eq. (1) in the form

$$u(x) = \sum_{k=1}^{\infty} u^{(k)}(x), \quad (3)$$

where

$$u^{(k)}(x) = \prod_{j=1}^n u_j^{(k)}(x_j). \quad (4)$$

The functions $u^{(k)}(x)$ can be found successively from the condition for minimizing certain functionals. Let $J(u, f)$ be a functional, minimizing which is equivalent to solving Eq. (1). (We shall consider several specific examples of such functionals below.) We construct a recursive procedure at whose k th step we shall seek a function $u^{(k)}(x)$ that minimizes the functional $J_k = J(u^{(k)}, f^{(k)})$, where

$$f^{(k)} = f - \sum_{i=1}^{k-1} Au^{(i)}. \quad (5)$$

When (2) is applied, the necessary conditions for minimizing J_k lead to a system of n one-dimensional Euler equations for the functions $u_j^{(k)}(x_j)$. These equations are nonlinear, but for the quadratic functionals J that are normally used each of them is linear in one of the functions $u_j^{(k)}(x_j)$. Each such equation can be considered separately and regarded as the condition for minimizing the functional J_k with respect to the function $u_j^{(k)}(x_j)$ with the other factors in $u^{(k)}(x)$ held fixed. Hence the cyclic-sequential solution of such equations relative to $u_j^{(k)}(x_j)$, $j = 1, \dots, n$, is equivalent to componentwise minimization of J_k .

The functionals J_k can also be minimized directly, for example, by any gradient method. When this is done, the components $z_j^{(k)}$ of the gradient will be the left-hand sides of the corresponding Euler equations.

Specific implementation. We consider the application of this scheme using the functionals most common in specific situations.

1. The operator A is self-adjoint and positive-valued. For simplicity we assume that the functions $f(x)$ and $u(x)$ are real-valued. In this case solving the problem (1) is equivalent to minimizing the functional

$$J(u, f) = \int_D (Au - 2f)u \, dx. \quad (6)$$

According to the general scheme of the method, at the k -th step it is necessary to minimize the functional

$$J_k = \int_D (Au^{(k)} - 2f^{(k)})u^{(k)} \, dx, \quad (7)$$

where the functions $u^{(k)}, f^{(k)}$ have the form (3) and (5) respectively.

To obtain the system of Euler equations for the functional (7), we find the partial variation $\delta_j J_k$ of the functional J_k with respect to the j -th factor $u_j^{(k)}$ in the representation (4) of the function $u^{(k)}$. To do this we write

$$J(u^{(k)} + \delta_j u^{(k)}, f^{(k)}) = J_k + \delta_j J_k + O((\delta_j u^{(k)})^2),$$

where

$$\delta_j u^{(k)} = \delta u_j^{(k)} \prod_{i \neq j} u_i^{(k)},$$

and $\delta u_j^{(k)}$ is a perturbation of the function $u_j^{(k)}$. As a result we obtain

$$\delta_j J_k = 2 \int_D \left(Au^{(k)} - f^{(k)} \right) \frac{u_j^{(k)}(x)}{u_j^{(k)}(x_j)} \delta u_j^{(k)}(x_j) dx.$$

Hence the Euler equations have the form

$$z_j^{(k)}(x_j) \equiv \int_{D_j} \frac{u_j^{(k)}(x)}{u_j^{(k)}(x_j)} \left(Au^{(k)} - f^{(k)} \right) d\hat{x}_j = 0, \quad j = 1, \dots, n. \quad (8)$$

It is easy to see that after cancelation in the first factor in the integrand the function $u_j^{(k)}$ remains in the j th equation only inside the symbol for the operator A , that is, the j th equation is linear in this function. It is obvious that the kind of the original equation is conserved.

To solve the system (8) one can use the following method of successive approximations. We choose some initial approximation of the functions $u_j^{(k)}$, $j = 2, \dots, n$, and solve the first equation of the system (8) for the function $u_1^{(k)}$. We fix the solution and solve the second equation of (8) for $u_2^{(k)}$, and so forth. After solving the n th equation, we must return to the first. This recursive procedure corresponds to componentwise minimization of the functional (7).

This scheme for solving the system of Euler equations is conserved also for other types of functionals.

2. The operator A is an arbitrary operator with the kernel $K \in l^2(D \times D)$, and the functions $u(x)$ and $f(x)$ are complex-valued. The problem (1) corresponds to the functional

$$J(u, f) = \int_D |Au - f|^2 dx. \quad (9)$$

The functional that is minimized at the k -th step has the form

$$J_k = \int_D |Au^{(k)} - f^{(k)}|^2 dx. \quad (10)$$

As in the preceding case, one can obtain the system of Euler equations

$$z_j^{(k)}(x_j) \equiv \int_{D_j} \frac{\bar{u}_j^{(k)}(x)}{\bar{u}_j^{(k)}(x_j)} A^* \left(Au^{(k)} - f^{(k)} \right) d\hat{x}_j = 0, \quad j = 1, \dots, n. \quad (11)$$

Here a bar over a function symbol denotes complex conjugation, and A^* is the adjoint of the operator A :

$$A^*v \equiv \int_D \bar{K}(x, x') v(x') dx' - \bar{\lambda} v(x). \quad (12)$$

The system (11) has the same properties as (8), that is, each of its equations is linear with respect to the corresponding function $u_j^{(k)}$.

3. Under the hypotheses of the preceding subsection one can also apply the functional

$$J(u, \bar{u}) = \int_D |u - \bar{u}|^2 dx, \quad (13)$$

where $\bar{u} = A^{-1}f$ is the unknown solution of Eq. (1).

It turns out that one can construct an algorithm for minimizing the functional (13) without inverting the operator A [15]. To do this it is necessary to apply another representation of the solution in which only the general form of (3) is preserved, and the individual terms have the form

$$u^{(k)}(x) = A^* \prod_{j=1}^n u_j^{(k)}(x_j). \quad (14)$$

At the k -th step the functional

$$J_k = \int_D |u^{(k)} - \tilde{u}^{(k)}|^2 dx \quad (15)$$

is minimized, where $\tilde{u}^{(k)}$ is the exact solution of the equation

$$A\tilde{u}^{(k)} = f^{(k)}. \quad (16)$$

The partial variation of this functional is

$$\delta_j J_k = 2 \operatorname{Re} \int_D (u^{(k)} - \tilde{u}^{(k)}) \overline{A^* [\delta u_j^{(k)} \prod_{i \neq j} u_i^{(k)}]} dx.$$

Applying the properties of the adjoint operator, formula (15), and the arbitrariness of the complex-valued function $\delta u_j^{(k)}$, we obtain the system of Euler equations

$$z_j^{(k)}(x_j) \equiv \int_{D_j} \frac{\tilde{u}^{(k)}(x)}{\tilde{u}_j^{(k)}(x_j)} (Au^{(k)} - f^{(k)}) d\hat{x}_j = 0, \quad j = 1, \dots, n. \quad (17)$$

It is easy to see that the characteristic properties of this system are similar to the properties of the system (8) and (11).

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