

Elastoplastic deformation of nonhomogeneous plates

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Received: 27 April 2010 / Accepted: 12 August 2010 / Published online: 19 October 2010
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Abstract Efficient projective-iterative schemes for the numerical implementation of the finite-element method are developed to calculate the stress and strain fields in elements of plate-and-shell systems, in particular in nonhomogeneous plates, under elastic and elastoplastic deformation. Rectangular plates with variously shaped (rectangular, circular, elliptic) holes in a plane stress state are considered. The effect of the elastoplastic deformation of the material on the stress and strain fields due to a slowly increasing load is studied. Schemes of successive approximations to account for plastic deformations in combination with projective-iterative schemes of numerical implementation of the finite-element method are constructed. The interaction of variously shaped holes is studied. The efficiency of projective-iterative variants of the finite-element method offering a several-fold reduction in running time is demonstrated. The proposed schemes are particularly attractive for the use of the finite-element method in numerical simulation of the nonlinear deformation of nonhomogeneous plate-and-shell systems. The numerical models considered can be used to good advantage in the study of the effect of other nonhomogeneities (geometrical shape irregularities, various inclusions, reinforcements). They can be used in fracture mechanics: failure starts with the development of discontinuities in variously shaped holes. The hole shape and dimensions can change during loading.

Keywords Elastoplastic deformation · Finite-element method · Holes · Projective-iterative schemes

1 Introduction

Alexey Antonovich Ilyushin contributed significantly to the mathematical theory of plasticity and other divisions of deformable solid mechanics. His general theory of plasticity allows one to investigate complex processes of the elastoplastic deformation of structural materials and elements [1, 2]. The computational methods based on this

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theory offer an efficient estimate of the stress and strain fields and the limiting state [3–12]. Structural elements employed in various branches of engineering, as a rule, are nonhomogeneous. The nonhomogeneity of a structure is characterized by a number of factors governing its structural features. For plate-and-shell systems among these features are geometrical imperfections, inclusions, and reinforcements of various types [13–16]. Holes are one of the basic types of nonhomogeneities [17–23]. The above-mentioned nonhomogeneities are due to working loads or manufacturing operations. Holes result in stress concentration, thus greatly increasing the risk of premature structural failure. Note that holes model discontinuities (cracks, cuts) in various models of fracture mechanics: failure starts with the development of such discontinuities, and during the process of loading the hole shape and dimensions change.

The variational-grid finite-element method (FEM) is an efficient numerical method for the solution of plate-and-shell structural-element deformation problems [24–26]. The projective-iterative schemes of numerical FEM implementation considered in this article offer a far (several times) shorter running time, which is of importance in numerical simulation of complex processes of nonlinear deformation of nonhomogeneous structural elements. As is known, the basic schemes of FEM simulation involve dividing the objects under study into variously shaped elements followed by the application of variational principles [27, Part B]. The application of the FEM results in high-order systems of linear equations with matrices of special form, for which direct methods are not sufficiently adequate. To improve the efficiency of the solution of such systems, it is expedient to use projective-iterative FEM-implementation schemes based on the apparatus of functional analysis [28, 29].

The necessity of accounting for elastoplastic deformations, which occur at high loading levels, greatly complicates the construction of FEM-implementation schemes. An efficient approach to attacking problems of elastoplastic deformation of structural elements involves the formulation of a series of elasticity problems with variable elastic parameters (modulus of elasticity, shear modulus, Poisson's ratio) and additional loads or deformations. These factors depend on the stress and strain fields in the structures, and we have nonlinear problems, which are solved by the method of successive approximations (in each approximation, the corresponding elasticity problem is solved). Ilyushin et al. [1–3, 6–8, 11] provided the basic ideas of such approaches. They have been implemented in the solution of a variety of problems of elastoplastic deformation and limiting states. Some aspects of the application of projective and iterative methods to plasticity problems are reported in [30]. With the above schemes of successive approximations, the solution of nonlinear problems proceeds until two successive approximations coincide to within a given accuracy. Projective-iterative FEM-implementation schemes speed up the convergence of successive approximations.

In this article, theoretically substantiated projective-iterative schemes of FEM implementation are proposed, and the elastoplastic deformation of plates with variously shaped (rectangular, circular, elliptic) holes is studied. Plastic zones are determined, and hole interaction is investigated. The above problems have been solved using developments in the theory of plasticity, many of which are due to Ilyushin.

Note that the first author of this article repeatedly participated as a member of the organizational committees and lecturer in scientific conferences, chaired by Professor A. A. Ilyushin.

2 Basics of projective-iterative methods

Among approximate methods for the solution of equations in normed spaces, iterative and projective methods occupy an important place [28, 29, 32], [31, Chaps. 5–9]. A combination of the ideas underlying these methods forms the basis of projective-iterative [33–35] and multigrid [36], [37, Chaps. 4–6] methods. Projective-iterative methods are efficient numerical methods for the solution of infinite-dimensional problems as demonstrated, for example, by their application to elliptic boundary problems. The development of these methods and their adaptation to various classes of problems in deformable solid mechanics is of great importance in terms of applications. Before proceeding to the construction of efficient projective-iterative FEM-implementation schemes, some results on the development and theoretical substantiation of solution methods for problems of conditional minimization of functionals in Hilbert spaces will be considered [38, 39].

2.1 Projective-iterative variant of the pointwise relaxation method

Let a bounded-below functional $F(u)$, $\inf_{u \in \Omega} F(u) = F^* > -\infty$, be defined on some set Ω of real Hilbert space H . Let it be desired to minimize $F(u)$ on Ω :

$$F(u) \rightarrow \inf, \quad u \in \Omega. \quad (1)$$

Consider the variant of the projective-iterative method of solution of the problem (1) which is based on the application of the pointwise relaxation method to the solution of a sequence of conditional minimization problems [38]. Let the functional $F(u)$ be approximated by a sequence of approximate functionals $\tilde{F}_n(\tilde{u}_n)$ ($n = N, N+1, \dots$) defined in finite-dimensional Hilbert spaces $\tilde{H}_n$ such that $\inf_{\tilde{u}_n \in \tilde{H}_n} \tilde{F}_n(\tilde{u}_n) = \tilde{F}_n^* > -\infty$. For each $\tilde{F}_n(\tilde{u}_n)$, the

following problem is formulated on $\tilde{\Omega}_n \subset \tilde{H}_n$:

$$\tilde{F}_n(\tilde{u}_n) \rightarrow \inf, \quad u \in \tilde{\Omega}_n. \quad (2)$$

Let $\tilde{H}_n$ be a vector space of dimension $N = N(n)$ with a Euclidean metrics. Let us specify an initial approximation $\tilde{u}_n^{(0)} = \{\tilde{u}_{1n}^{(0)}, \dots, \tilde{u}_{Nn}^{(0)}\}$ and construct k_n approximations $\tilde{u}_n^{(k)}$, ($k = 1, 2, \dots, k_n$) for the functional $\tilde{F}_n(\tilde{u}_n)$ on $\tilde{\Omega}_n \subset \tilde{H}_n$ using the pointwise relaxation method. We will take the last of them as an initial approximation to the solution of the problem of minimization of the functional $\tilde{F}_{n+1}(\tilde{u}_{n+1})$ on the set $\tilde{\Omega}_{n+1}$. Then we will obtain a sequence $\{\Phi_n^{-1}(\tilde{u}_n^{k_n})\}$ of approximations to the point of minimum u^* of the initial functional on Ω : $\tilde{u}_n^{(k+1)} = \{\tilde{u}_{1n}^{(k+1)}, \dots, \tilde{u}_{Nn}^{(k+1)}\}$. The quantities $u_{in}^{(k+1)}$ are determined from the solution of the inequality

$$\tilde{F}_n(\tilde{u}_{1n}^{(k+1)}, \dots, \tilde{u}_{i-1,n}^{(k+1)}, \tilde{u}_{in}^{(k+1)}, \tilde{u}_{i+1,n}^{(k)}, \dots, \tilde{u}_{Nn}^{(k)}) \leq \tilde{F}_n(\tilde{u}_{1n}^{(k+1)}, \dots, \tilde{u}_{i-1,n}^{(k+1)}, \tilde{v}_i, \tilde{u}_{i+1,n}^{(k)}, \dots, \tilde{u}_{Nn}^{(k)}) \quad (3)$$

for every $\tilde{v}_i(\tilde{u}_{1n}^{(k+1)}, \dots, \tilde{u}_{i-1,n}^{(k+1)}, \tilde{v}_i, \tilde{u}_{i+1,n}^{(k)}, \dots, \tilde{u}_{Nn}^{(k)}) \in \tilde{\Omega}_n$, $i = 1, 2, \dots, N$, we obtain $(\tilde{u}_n^{(k+1)}) = \Phi_{n+1} \Phi_n^{-1} \tilde{u}_n^{(k_n)}$, $k = 0, 1, \dots, k_n - 1$; $n = 1, 2, \dots$; $\tilde{u}_1^{(0)} \in \tilde{\Omega}_1$). Here, Φ_n is a linear operator that puts each element $u_n \in H_n$ in one-to-one correspondence with an element $\tilde{u}_n \in \tilde{H}_n$; Φ_n^{-1} is the operator of inverse mapping; the sets Ω and $\tilde{\Omega}$ are related as $\tilde{\Omega} = \Phi_n \Omega_n$.

Theorem 1 [39] Let functionals $F(u) \in C^1(\Omega)$ and $\tilde{F}_n(\tilde{u}_n) \in C^1(\tilde{\Omega}_n)$ be strongly convex [40], let a number $C > 0$ exist such that $\tilde{\chi}_n \geq C$, where $\tilde{\chi}_n$ is the constant of strong convexity of $\tilde{F}_n(\tilde{u}_n)$, and let the condition

$$\left| \tilde{F}_n(\tilde{u}_n) - F(\Phi_n^{-1} \tilde{u}_n) \right| \leq \tilde{\gamma}_n, \quad (4)$$

be satisfied for every $\tilde{u}_n \in \tilde{\Omega}_n$ with $\tilde{\gamma}_n \rightarrow 0$ when $n \rightarrow \infty$. Let the sequence of subspaces H_n be ultimately dense in H and let the sets Ω and $\tilde{\Omega}_n$ be related by the condition: for any $u \in \Omega$ there exists a sequence $\{\tilde{u}_n\}$, $\tilde{u}_n \in \tilde{\Omega}_n$ and $\lim_{n \rightarrow \infty} \Phi_n^{-1} \tilde{u}_n = u$, i.e., the point of minimum u^* is (Ω_n) -approximable [34].

Then the sequence $\{\Phi_n^{-1}(\tilde{u}_n^{(k_n)})\}$, where $\tilde{u}_n^{(k_n)}$ are defined by formula (3), converges to the point u^* of minimum of $F(u)$ on Ω . When the condition,

$$\left| \tilde{F}_n(\tilde{\Phi}_n u^*) - F(u^*) \right| \leq \tilde{\beta}_n, \quad \tilde{\beta}_n \rightarrow 0 \quad \text{when } n \rightarrow \infty,$$

is satisfied, the sequence $\{\tilde{u}_n^{(k_n)}\}$ converges to $\tilde{\Phi}_n u^*$, where the linear operator $\tilde{\Phi}_n$ is an extension of Φ_n to the space H .

2.2 Projective-iterative variant of the conjugate-gradient method

Let H be a real separable Hilbert space on which a bounded-above functional $F(u)$, $\inf_{u \in H} F(u) = F^* > -\infty$, belonging to the class $C^1(H)$ of functionals continuously differentiable on H is defined. Let it be desired to minimize $F(u)$ on H , i.e., to solve the problem (1) for $\Omega = H$. As in Sect. 2.1, we will obtain the sequence of

approximating problems (2) on $\tilde{\Omega}_n = \tilde{H}_n$. To solve them, use will be made of the iterative conjugate-gradient method [40]. In this case, only several approximations $\tilde{u}_n^{(k)}$ ($k = 1, 2, \dots, k_n - 1, k_n \leq K$, where K is a natural number) are constructed, and the last one is used as an initial approximation to the solution of the next, $(n + 1)$ -th, minimization problem. As a result, the sequence $\{\tilde{u}_n^{(k_n)}\}$ is obtained:

$$\begin{aligned} \tilde{u}_n^{(k+1)} &= \tilde{u}_n^{(k)} - \tilde{\alpha}_n^{(k)} \tilde{p}_n^{(k)}, \quad \tilde{u}_{n+1}^{(0)} = \Phi_{n+1} \Phi_n^{-1} \tilde{u}_n^{(k_n)}, \\ (k &= 1, 2, \dots, k_n - 1; n = N, N + 1, \dots; \tilde{u}_N^{(0)} \in \tilde{H}_N), \\ \tilde{p}_n^{(0)} &= \tilde{F}'_n(\tilde{u}_n^{(0)}), \quad \tilde{p}_n^{(k)} = \tilde{F}'_n(\tilde{u}_n^{(k)}) - \tilde{\beta}_n^{(k)} \tilde{p}_n^{(k-1)}, \\ (k &= 1, 2, \dots, k_n; n = N, N + 1, \dots), \end{aligned} \quad (5)$$

where the quantities $\tilde{\alpha}_n^{(k)}, \tilde{\beta}_n^{(k)}$ are chosen from the conditions

$$\tilde{g}_n^{(k)}(\tilde{\alpha}_n^{(k)}) = \min_{\alpha \geq 0} \tilde{g}_n^{(k)}(\alpha), \quad \tilde{g}_n^{(k)}(\alpha) \equiv \tilde{F}_n(\tilde{u}_n^{(k)} - \alpha \tilde{p}_n^{(k)}) \quad (6)$$

and

$$\tilde{\beta}_n^{(k)} = \begin{cases} \left(\tilde{F}'_n(\tilde{u}_n^{(k)}), \tilde{F}'_n(\tilde{u}_n^{(k-1)}) - \tilde{F}'_n(\tilde{u}_n^{(k)}) \right) / \left\| \tilde{F}'_n(\tilde{u}_n^{(k-1)}) \right\|^2, & k \in I_1^n \\ 0, & k \in I_2^n \end{cases} \quad (7)$$

and the index sets I_1^n and I_2^n are such a that $I_1^n \cup I_2^n = \{0, 1, 2, \dots, k_n\}$, $0 \in I_2^n$.

Depending on the choice of the sets I_1^n and I_2^n , different projective-iterative variants of the conjugate-gradient method can be obtained.

Let the functionals $F(u)$ and $\tilde{F}_n(\tilde{u}_n)$ satisfy the proximity condition (4) for every $\tilde{u}_n \in \tilde{H}_n$ with $\tilde{\gamma}_n \rightarrow 0$ when $n \rightarrow \infty$ ($\sum_{n=N}^{\infty} \tilde{\gamma}_n < \infty$), and the condition

$$\left\| \tilde{F}'_n(\tilde{u}_n) - \Phi_n F'_n(\Phi_n^{-1} \tilde{u}_n) \right\| \leq \tilde{\delta}_n \quad \text{with } \tilde{\delta}_n \rightarrow 0 \quad \text{when } n \rightarrow \infty; \quad (8)$$

let the sequence of subspaces H_n be ultimately dense in H , and let the spaces H and $\tilde{H}_n$ be related by the following condition (A): for any $u \in H$ there exists a sequence $\{\tilde{u}_n\}$, $\tilde{u}_n \in \tilde{H}_n$ such that $\lim_{n \rightarrow \infty} \Phi_n^{-1} \tilde{u}_n = u$.

Theorem 2 Let the functionals $F(u) \in C^1(H)$ and $\tilde{F}_n(\tilde{u}_n) \in C^1(\tilde{H}_n)$ be strongly convex, and let the Lipschitz condition [40] for the gradient $F'(u)$

$$\|F'(v) - F'(u)\| \leq L\|v - u\|,$$

$u, v \in H$, $L = \text{const} > 0$, the condition (A), and the proximity conditions (4), (8) be satisfied.

Then, with any choice of the index sets I_1^n, I_2^n , $0 \in I_2^n$ and any initial approximation $\tilde{u}_N^{(0)} \in \tilde{\Omega}_N$ for the sequence $\{\tilde{u}_n^{(k_n)}\}$ that satisfies the conditions (5) to (7) the following estimates hold

$$\begin{aligned} \left\| \Phi_n^{-1} \tilde{u}_n^{(k_n)} - u^* \right\|^2 &\leq 2\eta^{-1} \left(\tilde{\gamma}_n + \left(\tilde{F}_n(\tilde{u}_n^{(0)}) - F^* \right) q_n^{(k_n)} \right), \\ 0 \leq F\left(\Phi_n^{-1} \tilde{u}_n^{(k_n)}\right) - F(u^*) &\leq \eta^{-1} \left\| F'\left(\Phi_n^{-1} \tilde{u}_n^{(k_n)}\right) \right\|^2, \\ \left\| \Phi_n^{-1} \tilde{u}_n^{(k_n)} - u^* \right\|^2 &\leq \eta^{-1} \left\| F'\left(\Phi_n^{-1} \tilde{u}_n^{(k_n)}\right) \right\|, \end{aligned}$$

where

$$q_n = (1 - \eta^3) / (2L_n(\eta^2 + L_n^2)), \quad L_n = 2\delta_n + CL, \quad 0 < q_n < 1,$$

and η is the constant of strong convexity of the functional F .

2.3 Projective-iterative FEM implementation

The basic idea of the projective-iterative FEM variant is that the original extremum problem (1) is approximated, using the FEM, by a sequence of discrete extremum problems ($n = 1, 2, \dots$) for multivariable functions [28, 38]. Each of the problems, starting with some number $n = N$, is solved by the method of successive overrelaxation using the procedure described in [41], according to which the system stiffness matrix need not be stored in explicit form, and the values of the displacements u and v at the current grid point (i, j) are expressed in terms of the grid-point values of the displacements of the four finite elements adjacent to the grid point (i, j) [42]. For problems where stress concentration occurs, it is advisable to use adaptive grids.

2.3.1 Construction of an adaptive grid

To construct an adaptive grid from rectangular finite elements, the following algorithm is used:

- (1) in the region ω under consideration, a square grid of spacing $h_n > 0$ ($n = N$) formed by the straight lines $x_i = ih_n$, $y_j = jh_n$, $0 < i < N_x$, $0 < j < N_y$, $N_x = 2a/h_n$, $N_y = 2b/h_n$ is introduced (Fig. 1a);
- (2) a zone of assumed stress concentration of length l_1 and width l_2 is specified (Fig. 1a);
- (3) an integer $d > 0$ is specified, and the zone of assumed stress concentration is partitioned by straight lines parallel to the Ox and Oy axes spaced by distance $h_n^d = h_n/d$ (Fig. 1b);
- (4) as a result of these operations, we obtain a adaptive finite-element grid of dimension

$$(N_x + 1 + (d - 1)C_x) \times (N_y + 1 + (d - 1)C_y), \quad (9)$$

where C_x and C_y are the numbers of elements in the stress-concentration zone along the Ox and Oy axes, respectively;

- (5) a sequence of nested adaptive grids is obtained by triangulating the finite elements along the Ox and Oy axes with step $h_{n+1} = h_n/2$ outside the stress concentration zone and with step $h_{n+1}^d = h_n^d/2$ therein.

Thus, on each of finite-element grid we will obtain a set of finite elements of different dimensions: a square finite element of type I, a square finite element of type II whose sides are d times smaller than those of the element of type I, and a rectangular finite element of type III whose side ratio is equal to d (Fig. 1b).

The problem algorithm based on the application of the projective-iterative FEM variant on the sequence of adaptive grids is as follows:

- (1) for given geometrical parameters of the problem and a given computational accuracy ε , relaxation factor ω , and initial grid spacing h_n ($n = N$), an initial adaptive grid of the dimension (9) is constructed;

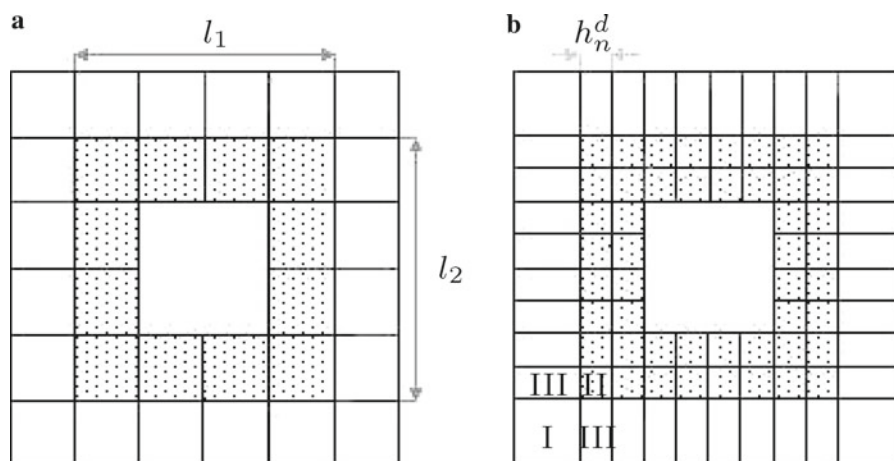


Fig. 1 Construction of adaptive grid

- (2) an initial approximation is specified for $n = N$;
- (3) the functional of the problem (1) is approximated by a multivariable function, which is minimized using the method of successive overrelaxation;
- (4) the number k_n of the approximations to be constructed in the n -th step of the projective-iterative process is chosen equal to the least integer k that satisfies the inequality $\|z_n^{(k)} - z_n^{(k-1)}\| \leq \varepsilon$, where $z_n^{(k)}$ is the approximate solution of the n -th finite-dimensional problem;
- (5) the solution obtained on the n -th adaptive grid is interpolated to a finer $(n+1)$ -th adaptive finite-element grid and used as an initial approximation to the point of minimum for the corresponding multivariable function (go to step 3).

2.3.2 Numerical analysis

The proposed modifications of projective-iterative methods have been applied to a plane, a three-dimensional, and a contact-elasticity problem [38]. In the solution of the three-dimensional problem for a multilayer pile and the contact problems, the formulations given in [41, 43, 44] were used. Presented below are the results for the model problem [38, 45]. The stress and strain fields of an elastic isotropic plate with a square hole at the center under a uniform compressive load are studied (Fig. 2a). The plate dimensions are $L_1 = 0.06$ m and $L_2 = 0.08$ m, the hole measures 0.02×0.02 m, the modulus of elasticity is $E = 3300$ MPa, and Poisson's ratio is $\nu = 0.34$. The values of E and ν are the same as those for the plate material in the experiments conducted using the photo-elasticity method [45].

The projective-iterative FEM variant was implemented on a sequence of finite-element grids made up of rectangular four-point Lagrangian elements with a displacement calculation accuracy of 10^{-6} : $|u_{n+1} - u_n| \leq \varepsilon$ ($\varepsilon = 10^{-6}$), each grid being constructed from the previous one by dividing a grid element into four identical ones (13×17 , 25×33 , 49×65 , 97×129 , 193×257). The calculations were carried out for zero initial approximation specified on the coarse set (13×17). In terms of running time, the projective-iterative FEM variant proved to be 5 times more efficient than the traditional FEM. The numerical values of the stresses and strains are in good agreement with the experimental data obtained by the photo-elasticity method, the difference being within 3%. The distribution of the principal stresses σ_1 , σ_2 determined from σ_x , σ_y and σ_{xy} [46, Chap. 2] is shown in Fig. 2b and c.

In the problem of the stress and strain fields of a pile of elastic layers under a normal load distributed over the top surface, the approach proposed in [41] was used, and the efficiency of the projective-iterative FEM variant on a sequence of finite-element grids was studied. The running time is more than 4 times shorter in comparison with the traditional FEM [47]. For the contact problems with ideal unilateral constraints [43, 44], the projective-iterative scheme based on the conjugate-gradient method and the method of successive overrelaxation offers a more than 80-fold reduction in running time in comparison with the usual FEM.

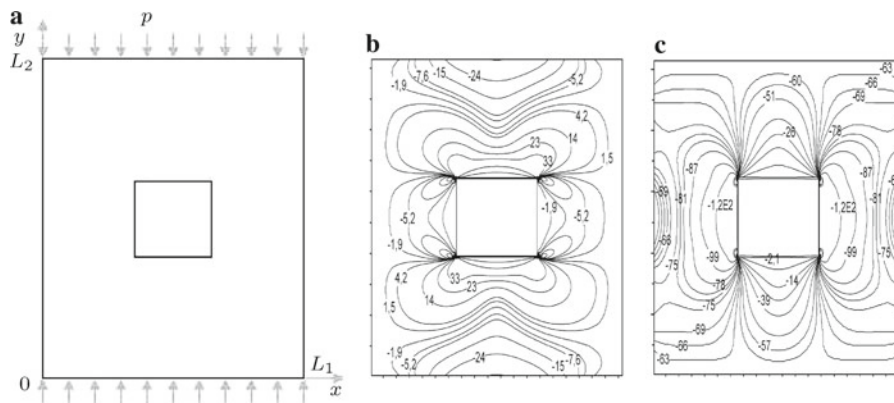


Fig. 2 Loading pattern and lines of principal stresses σ_1 and σ_2

It is of interest to study the effect of the shape of a finite element on the computational efficiency of the projective-iterative FEM-implementation scheme. We consider linear and bilinear approximations in the projective-iterative process (accordingly triangular and rectangular finite elements) in the problem of determining the stress and strain fields of a thin elastic plate with a rectangular hole at given displacements on its boundary which vary linearly [48]. Projective-iterative FEM schemes for rectangular and triangular finite elements are developed, and the efficiencies of the corresponding computational algorithms are compared. The use of rectangular finite elements in the projective-iterative process reduces the running time (by a factor of about 2) in comparison with the use of triangular finite elements, and the stresses obtained using the above types of finite elements differ from one another by no more than 1.7%. The obtained stress fields are in good agreement with those obtained by the finite-difference method [49, Chap. 4]. In determining the stress and strain fields, it is expedient to use rectangular finite elements. However, at stress concentration sites (in the vicinity of nodal points) more accurate values for the normal stresses σ_x , σ_y , and σ_{xy} , the tangential stresses, are obtained with rectangular and triangular elements, respectively. The same qualitative results are obtained for a circular hole [42].

3 Solution methods for elastoplastic problems

Let us give a brief characteristic of solution methods for elastoplastic problems [1, 8, 11, 12].

In the method of variable elastic parameters, an iterative process is constructed, and in each approximation an elasticity problem with variable modulus of elasticity $\bar{E}$, shear modulus $\bar{G}$, and Poisson's ratio $\bar{\nu}$ is solved. In deformation theory, the following relations hold

$$\varepsilon_{ij} = \frac{1}{\bar{E}} ((1 + \bar{\nu}) \sigma_{ij} - 3\bar{\nu} \sigma \delta_{ij}),$$

$$\bar{E} = 2\bar{G}(1 + \bar{\nu}), \quad \bar{G} = \frac{1}{2\psi}, \quad \bar{\nu} = \frac{E\psi - 1 + 2\nu}{2E\psi + 1 - 2\nu}, \quad \psi = \frac{3\varepsilon_i}{2\sigma_i}, \quad (10)$$

where σ_{ij} and ε_{ij} are the stress and strain tensors, σ_i and ε_i are the stress and strain intensities, E and G are the modulus of elasticity and shear modulus of the material, and ν is Poisson's ratio.

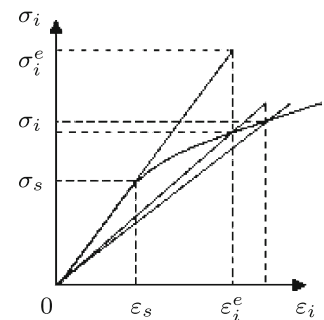
In a first approximation, we set $\bar{E}_1 = E$, $\bar{\nu}_1 = \nu$ and determine σ_{ij1} , σ_{i1} , ε_{i1} , ψ_1 . In a second approximation, we determine $\bar{E}_2 = \sigma_{i1}/\varepsilon_{i1}$, $\bar{\nu}_2$ (10) at ψ_1 , and so on. The process is run until two successive approximations coincide to within a given accuracy, and the points determining σ_{im} and ε_{im} should be close to the deformation curve, $\bar{E}_m \approx \bar{E}_{m-1}$. The scheme of successive approximations with the use of the material-deformation curve is shown in Fig. 3. The method of successive loadings [50] is a variant of this method.

In the method of additional loads, additional quantities F_i^0 and p_i^0 are added to the external loads F_i and p_i in the equilibrium equations and boundary conditions:

$$\sigma_{ij,j}^e + F_i + F_i^0 = 0; \quad \sigma_{ij} n_j = p_i + p_i^0,$$

$$F_i^0 = - \left[\left(1 - \frac{1}{2G\psi} \right) s_{ij}^e \right]_{,j}; \quad p_i^0 = \left(1 - \frac{1}{2G\psi} \right) s_{ij}^e n_j, \quad (11)$$

Fig. 3 Scheme of successive approximations



where s_{ij} are stress deviators, and the superscript “e” denotes the quantities for an elastic body whose deformations are the same as in the elastoplastic body.

In a first approximation, we set $F_i^0 = p_i^0 = 0$ and solve the elasticity problem to determine σ_{ij1}^e and σ_{i1}^e (on the deformation curve, σ_{i1} and ε_{i1} correspond to σ_{i1}^e). Then the first-approximation parameter ψ_1 and stresses σ_{ij1}^e are determined as

$$\psi_1 = \frac{3\varepsilon_{i1}}{2\sigma_{i1}} = \frac{(1+\nu)\sigma_{i1}^e}{E\sigma_{i1}}, \quad \sigma_{ij1} = \frac{\sigma_{ij1}^e}{2G\psi_1} + \sigma^e \left(1 - \frac{1}{2G\psi_1}\right) \delta_{ij}. \quad (12)$$

In a second approximation, we solve the elasticity problem with the loads F_i^0 and p_i^0 (11), where at $\psi = \psi_1$, $s_{ij}^e = s_{ij1}$ and further calculations follow the scheme: $\sigma_{ij2}^e \rightarrow \sigma_{i2}^e \rightarrow \varepsilon_{i2}^e \rightarrow \sigma_{i2} \rightarrow \psi_2 \rightarrow F_{i2}^0, p_{i2}^0$. In a third approximation, the elasticity problem with the additional loads F_{i2}^0 and p_{i2}^0 is solved, and so on. The process is run until two successive approximations coincide to within a given accuracy: $\sigma_{ijm} \approx \sigma_{ij(m-1)}$. Note that it was the method of additional loads that was first considered by Ilyushin [1] and named by him “method of elastic solutions”.

For the method of additional strains, the following relations hold:

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^0, \quad \varepsilon_i = \varepsilon_i^e + \varepsilon_i^0, \quad \varepsilon_{ij}^0 = \frac{E\psi - 1 - \nu}{E} s_{ij}, \quad (13)$$

where the superscript “e” means that ε_{ij} corresponds to a fictitious elastic body whose stresses σ_{ij} are the same as in the elastoplastic body, and ε_{ij}^0 are the additional strains.

In a first approximation, we set $\psi = 1/(2G)$, $\varepsilon_{ij}^0 = \varepsilon_i^0 = 0$ and determine $\sigma_{ij1}^e, \varepsilon_{ij1}^e, \varepsilon_{i1}^e, \sigma_{i1}, \psi_1 = 3\varepsilon_{i1}/(2\sigma_{i1})$. In a second approximation, the elasticity problem with the strains ε_{ij}^0 determined from (13) at ψ_1 is solved, and further calculations follow the scheme: $\varepsilon_{ij2}^e \rightarrow \varepsilon_{i2}^e \rightarrow \sigma_{i2} \rightarrow \psi_2 \rightarrow \varepsilon_{ij2}^0$. The process is run until two successive approximations coincide to within a given accuracy.

The above descriptions of the methods relate to deformation plasticity theory [1, 11]. These methods are also developed for flow theory [8]. It is known that for simple loading (the stress deviators vary in proportion to a single parameter) the solutions of elastoplastic problems on the basis of the two theories coincide [1–3, 7, 11].

At stress concentration sites in the vicinity of holes, complex loading takes place [2, 7]. In this case, it is expedient to use adaptive grids [42, 47], which allow one to account for the local features of loading.

4 Problems of elastoplastic deformation of plates with holes

Consider problems concerning the determination of the stress and strain fields in rectangular plates with variously shaped holes with due regard given to the elastoplastic deformation of the material using both the projective-iterative FEM-implementation schemes developed and the traditional FEM.

4.1 Plate with a rectangular hole

Consider a rectangular plate with a rectangular hole at the center. Figure 4a depicts the plate-loading diagram (because of the symmetry of the problem, only a quarter of the plate is shown). We will apply the variational method [7, 27] using the method of variable elastic parameters [8, 12] in the solution of the elastoplastic deformation problem. As a result, we will obtain an iterative process of successive approximations. In each approximation, a nonhomogeneous elasticity problem is solved, and the following functional is minimized [51, 52]

$$\Pi = \int_{-a}^a \int_{-b}^b \left[\frac{\bar{G}}{1-\bar{\nu}} \left(\frac{\partial u}{\partial x} \right)^2 + \frac{\bar{G}}{1-\bar{\nu}} \left(\frac{\partial v}{\partial y} \right)^2 + \frac{2\bar{G}\bar{\nu}}{1-\bar{\nu}} \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\bar{G}}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right] dx dy - \int_{-a}^a p v dx, \quad (14)$$

where $\bar{G}, \bar{\nu}$ are the variable shear modulus and Poisson’s ratio, which depend on the stress and strain intensities σ_i and ε_i in the plate, and u, v are the displacements.

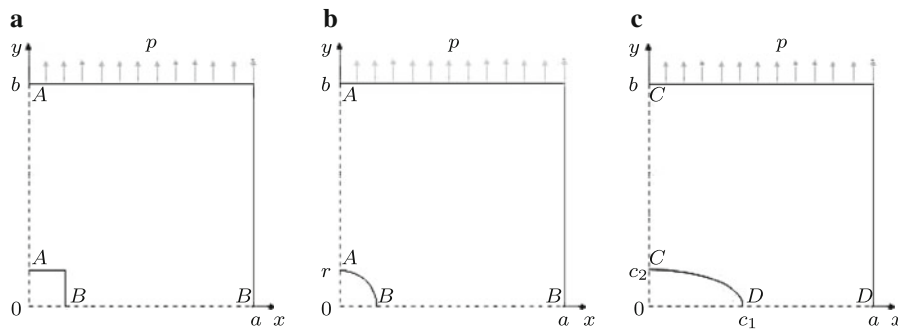


Fig. 4 Loading pattern

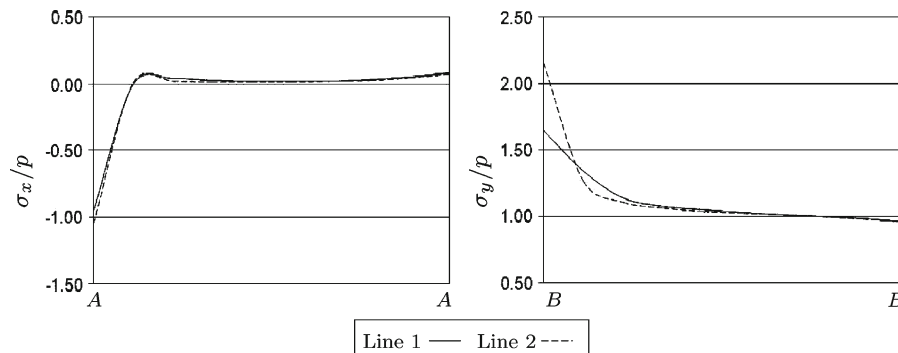


Fig. 5 Distribution of the stresses σ_x and σ_y

The strain intensity is calculated as

$$\varepsilon_i = \frac{2}{3} \left(\frac{1 - \bar{\nu} + \bar{\nu}^2}{(1 - \bar{\nu})^2} (\varepsilon_x + \varepsilon_y)^2 - 3\varepsilon_x \varepsilon_y + \frac{3}{4} \gamma_{xy}^2 \right)^{\frac{1}{2}}, \quad (15)$$

where ε_x , ε_y , γ_{xy} are the plate strains.

The scheme of successive approximations for the method of variable elastic parameters is shown in Fig. 3. In each approximation, the elasticity problems are solved using the projective-iterative FEM-implementation schemes developed. Below are the results for a square plate composed of 7005 aluminium alloy ($\sigma_s = 200$ MPa) with dimensions $a = b = 0.1$ m, the sides of the square hole being equal to 0.02 m.

The calculation of the stress and strain fields for the elastoplastic deformation of the plate with a rectangular (square) hole at the basis of the computational schemes developed offers a 45-fold reduction in running time in comparison with the traditional FEM.

The solution of the elastoplastic problem obtained in [51] using the finite-difference method differs from the solution based on the projective-iterative method by no more than 3.1%. The theoretical solution for the elasticity problem is in good agreement with the experimental data obtained using the photo-elasticity method.

4.2 Plate with a circular hole

Consider a rectangular plate with a circular hole at the center. Figure 4b shows the plate-loading diagram. The plate material and dimensions are the same as in Sect. 4.1, and the radius of the circular hole is $r = 0.01$ m (the diameter is equal to the side of the square hole).

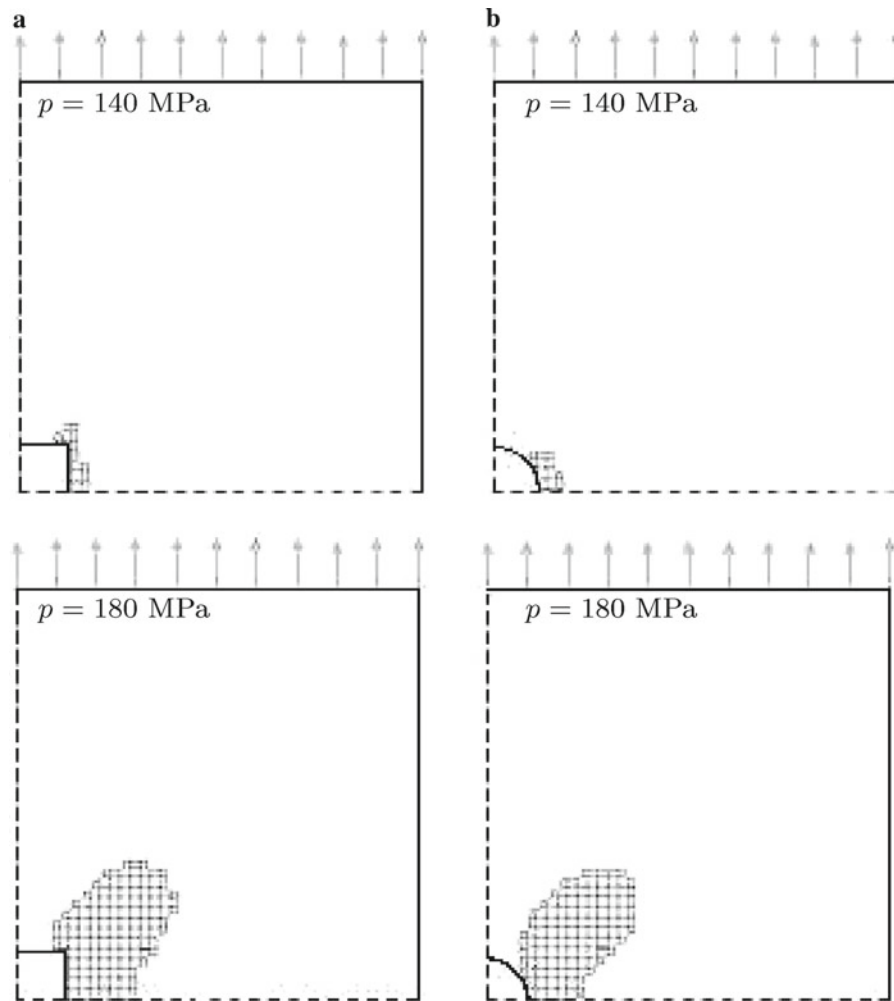


Fig. 6 Plastic zones in plate with square hole and circular hole

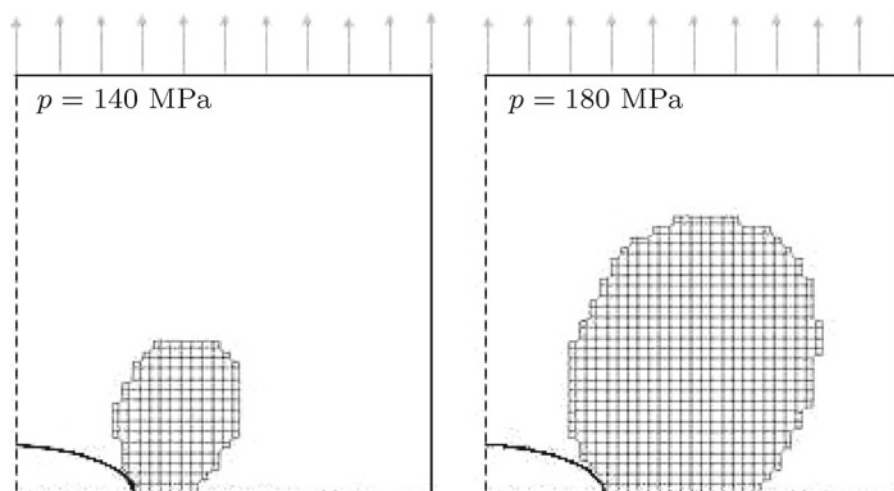


Fig. 7 Plastic zones in plate with elliptic hole

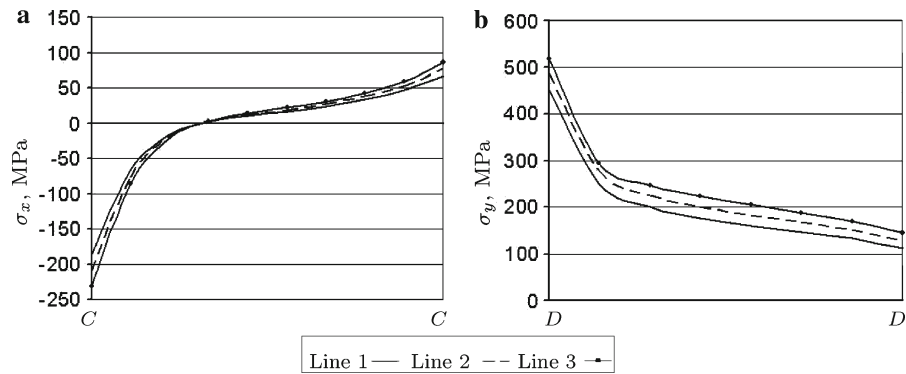


Fig. 8 Distribution of the stresses σ_x and σ_y

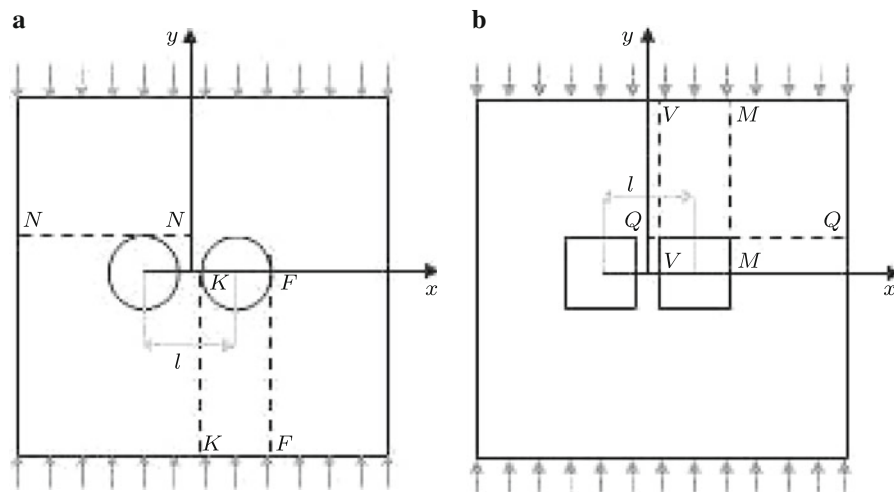


Fig. 9 Loading diagram and holes position

The elastoplastic deformation is also calculated using the variational method together with the method of variable elastic parameters. The scheme of successive approximations is shown in Fig. 3. Equations (14) and (15) are used. Figure 5 gives the distribution of the stresses σ_x and σ_y at the plate sections shown in Fig. 4b.

The calculation of the stress and strain fields for the elastoplastic deformation of the plate with the circular hole on the basis of the computational schemes developed offers a 24-fold reduction in running time in comparison with the traditional FEM.

For the elasticity problems, the solutions obtained using the projective-iterative FEM-implementation schemes are compared with the known analytical solutions on the basis of the calculus of complex variables with the use of Cauchy-type integrals [53]. The difference is within 3%. According to the recommendations made in [17], the hole-to-plate dimension ratio was chosen such that the results differ only slightly from those for an infinite strip presented in [17, 53].

Figure 5 gives the distribution of the dimensionless stresses σ_x/p and σ_y/p , respectively. Lines 1 correspond to the plate with the square hole, and lines 2 correspond to the plate with the circular hole. Figure 6 shows the plastic zones for different loads, (a) the plate with the square hole, (b) the plate with the circular hole.

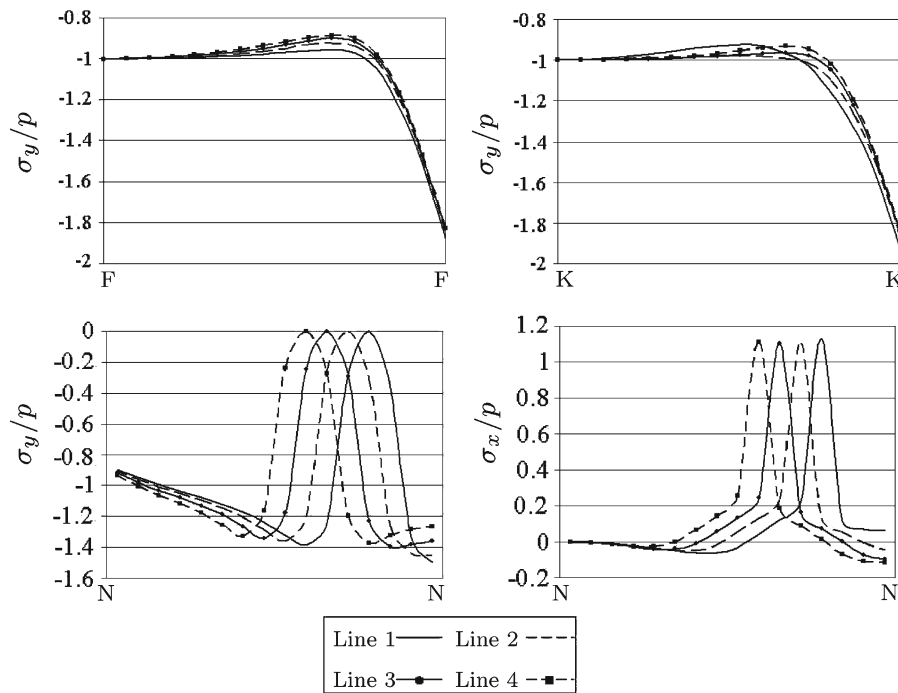


Fig. 10 Distribution of the stresses σ_x and σ_y in plate with two circular holes

4.3 Plate with an elliptic hole

Consider a rectangular plate with an elliptic hole at the centre. The plate-loading diagram is shown in Fig. 4c. The plate material and dimensions are the same as in Sect. 4.1, and the semi-axes of the ellipse are $c_1 = 0.03$ m and $c_2 = 0.01$ m. The problem is solved using the projective-iterative FEM-implementation schemes. Figure 7 shows the plastic zones for different compressive loads. Figure 8 gives the distribution of the stresses σ_x and σ_y (lines 1 correspond at $p = 140$ MPa, lines 2 correspond at $p = 160$ MPa, lines 3 correspond at $p = 180$ MPa) at the plate sections shown in Fig. 4c.

The calculation of the stress and strain fields for the elastoplastic deformation of the plate on the basis on the computational schemes developed offers a 32-fold reduction in running time in comparison with the traditional FEM.

The obtained solutions are in good agreement with the known analytical solutions reported in [53, Chaps. 2 and 5], [54]. The difference is within 2.8%, thus confirming the adequacy of the algorithms developed. The plate dimensions were chosen from the same considerations as for the plate with the circular hole [54].

A comparison with the calculated results for the plate with the circular hole, whose radius is equal to the minor semi-axis of the ellipse, shows that for the plate with the elliptic hole the projective-iterative FEM-implementation schemes are more efficient.

4.4 Plate with two circular and rectangular holes

Of considerable interest is the study of the stress and strain fields in a plate with two or more holes. The construction of plastic zones for various hole dimensions and various hole spacings allows one to study hole interaction.

Consider the problem of determining the stress and strain fields for a plate with two identical holes (circular or rectangular). The loading diagram and the hole position are shown in Fig. 9.

As in the problems considered above, the variational method is used, and the method of variable elastic parameters is applied to account for plastic deformations. In each approximation of the iterative process, a nonhomogeneous

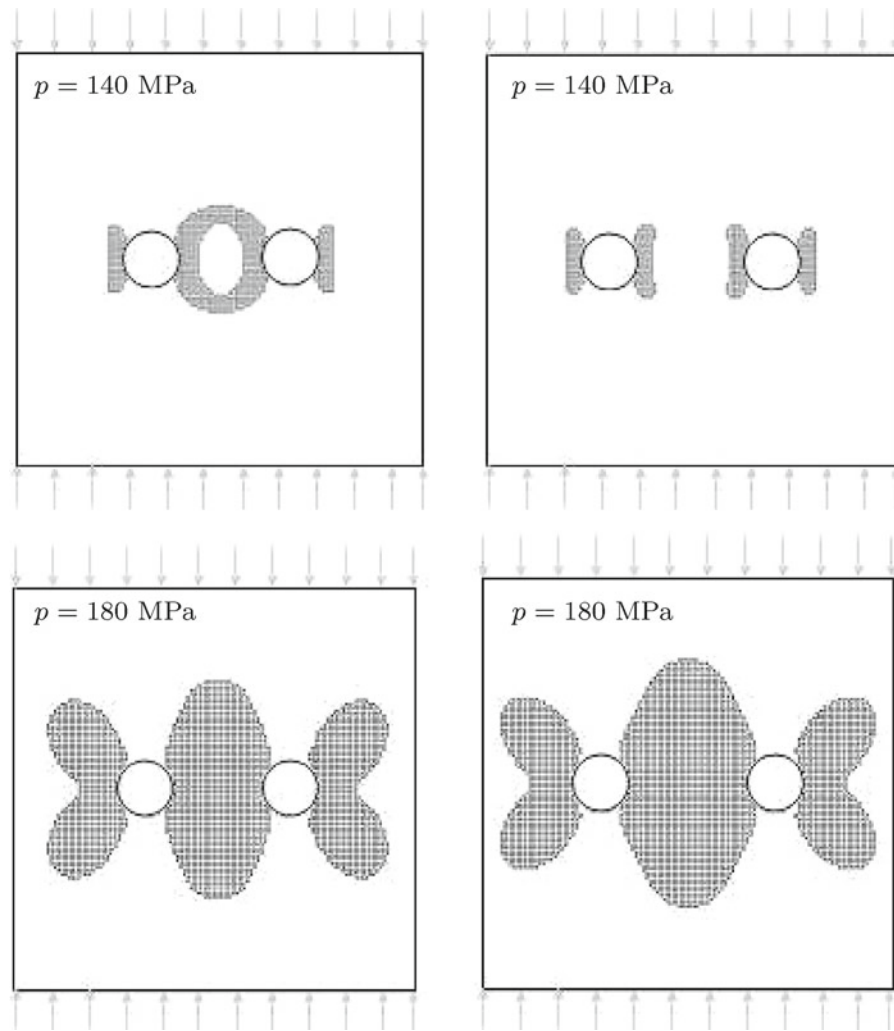


Fig. 11 Plastic zones in plate with two circular holes

elasticity problem is solved, and the functional (14) is minimized. The strain intensity is calculated from Eq. 15. The scheme of successive approximations is shown in Fig. 3.

Below are the results of the numerical analysis.

The analysis involved $a = b = 0.15$ m plates made of 7005 aluminium alloy with $\varnothing 0.04$ m circular holes and 0.04×0.04 m square holes. The compressive loads responsible for plastic deformation were varied from 140 MPa to 180 MPa. The study shows that, at such compressive loads, loss of stability for plates does not occur [10, 12, 21, 22]. The projective-iterative FEM scheme was implemented on a sequence of uniform finite-element grids (31×31 , 61×61 , 121×121 , 241×241 , 481×481 , 921×921 , 1841×1841) and adaptive grids (37×43 , 73×85 , 145×169 , 289×337 , 577×673 , 1153×1345 , 2305×2689) made up of rectangular four-point elements with computational accuracy $\varepsilon = 10^{-5}$.

For a plate with two circular holes, $p = 180$ MPa, $l = 0.12$ m, a projective-iterative FEM scheme on the sequence of seven adaptive grids offers a 30-fold reduction in running time in comparison with the traditional FEM. For other values of p and l the projective-iterative schemes are more efficient.

The analytical solution by the multipole expansion method [55] differs from the solution on the basis of the projective-iterative FEM-implementation scheme by no more than 3.5%.

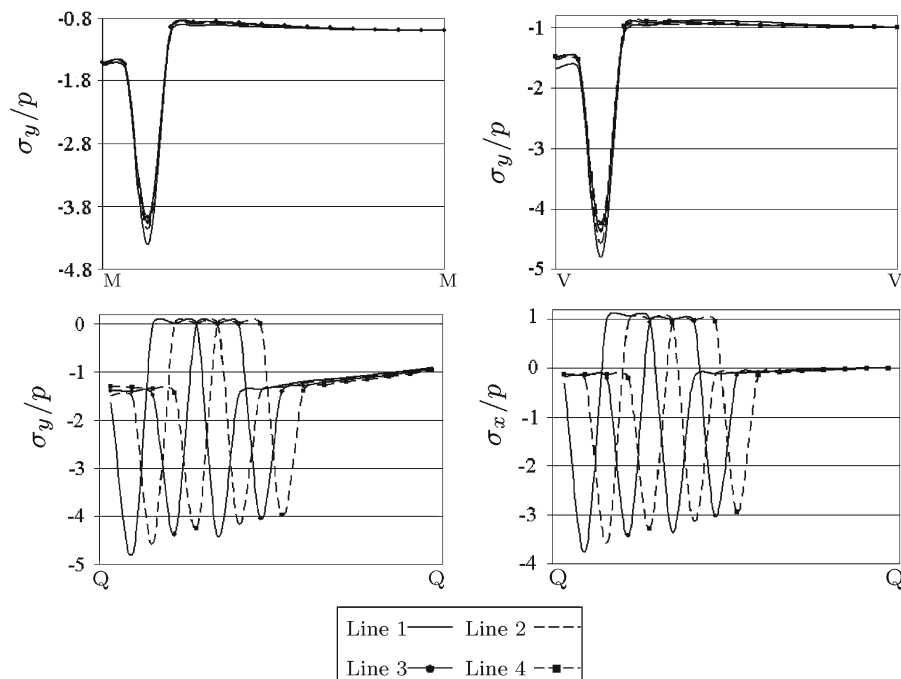


Fig. 12 Distribution of the stresses σ_x and σ_y in plate with two square holes

Figure 10 gives the distribution of the dimensionless stresses σ_x/p and σ_y/p for a plate with two circular holes, whose centers are separated by different distances, at the plate sections shown in Fig. 9a. Lines 1 to 4 correspond to distances l equal to 0.06 m, 0.08 m, 0.1 m, and 0.12 m. Figure 11 shows the plastic zones for two circular holes with $l = 0.1$ m (on the left) and $l = 0.12$ m at $p = 140$ MPa and $p = 180$ MPa.

Figure 12 gives the distribution of the dimensionless stresses σ_x/p and σ_y/p for a plate with two square holes, whose centers are separated by different distances, at the plate sections shown in Fig. 9b. Lines 1 to 4 correspond to distances l equal to 0.06 m, 0.08 m, 0.1 m, and 0.12 m. Figure 13 shows the plastic zones for two rectangular holes with $l = 0.1$ m (on the left) and $l = 0.12$ m. A comparison of Figs. 11 and 13 show that for plastic deformation hole interaction is qualitatively the same both for circular and square holes.

Note that similar projective-iterative FEM-implementation schemes can be constructed for shallow shells with variously shaped holes.

5 Conclusions

In this paper, projective-iterative schemes for numerical FEM implementation have been proposed. The application of these schemes to the solution of plane problems of the elastoplastic deformation of nonhomogeneous rectangular plates with variously shaped (rectangular, circular, elliptic) holes demonstrates their high efficiency in terms of running-time saving in comparison with the traditional FEM; in the problems considered, their use offers a 4- to 80-fold reduction in running time. This is of great importance in numerical simulation of the deformation and failure of plate-and-shell structural elements, where one has to solve a large number of similar problems using the FEM.

When triangular elements (linear approximations) are used, projective-iterative FEM variants give more accurate values of the stresses and strains in the case of stress concentration. However, their use results in a longer running time in comparison with rectangular elements (bilinear approximations).

Elastoplastic deformation is accounted for by the construction of some processes of successive approximations. In each approximation, a nonhomogeneous elasticity problem is solved using projective-iterative FEM-implementation schemes. These schemes are efficient in the study of the elastoplastic deformation of plate-and-shell systems with

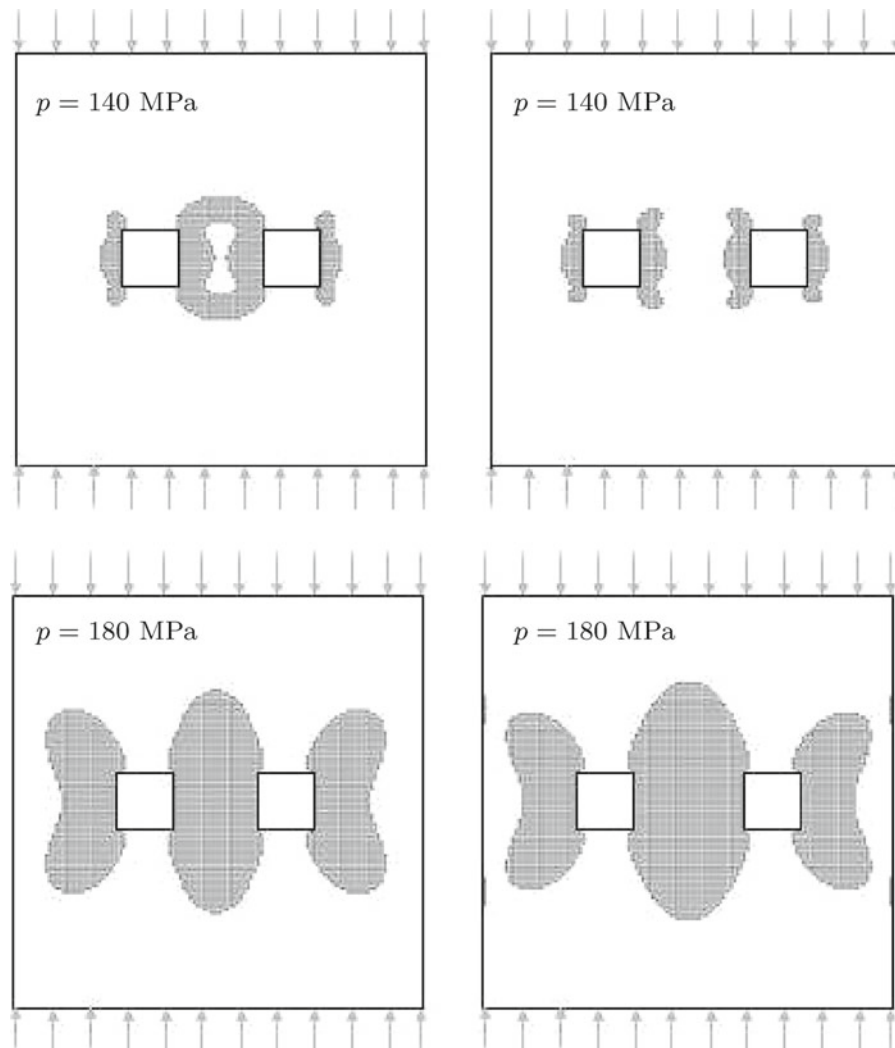


Fig. 13 Plastic zones in plate with two square holes

holes and other nonhomogeneities. The numerical simulation schemes developed and the computational algorithms implemented allow one to study hole interaction and construct plastic zones.

The efficiency of the projective-iterative FEM-implementation schemes increases with the dimension of the finite-dimensional problems to be solved (by a factor of up to several tens).

The proposed approach can be successfully applied to problems of fracture mechanics: holes (discontinuities) model the initial stage of structural failure.

The projective-iterative schemes considered can also be used in simulating the elastoplastic deformation and failure of elements of plate-and-shell systems with other types of nonhomogeneities such as various geometrical imperfections, inclusions, and reinforcements.

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