

ON THE RELATIONSHIP BETWEEN PROPERTIES OF SOLUTIONS OF DIFFERENCE EQUATIONS AND THE CORRESPONDING DIFFERENTIAL EQUATIONS

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We establish conditions under which the existence of a periodic solution of a differential equation is preserved if a solution of the corresponding difference equation possesses the same property. We prove the convergence of periodic solutions of a system of difference equations to a periodic solution of a system of differential equations. Analogous problems are considered for bounded solutions.

1. Introduction

An efficient method for the investigation of differential equations is based on the transition from differential equations to difference equations obtained from the differential equations by replacing the derivative by the corresponding difference relation. Difference equations are easily solved by a stepwise method, especially if modern computers are used. However, this method can be used for the solution of these equations only on a finite interval (at finitely many nodes). As a result, this method does not give qualitative characteristics (boundedness, periodicity, and stability) of the corresponding solutions of differential equations. For this reason, the problem of the investigation of the qualitative correspondence between solutions of differential equations and solutions of difference equations is of major importance.

Conditions for the preservation of the properties of periodicity, stability, and oscillation of solutions of difference equations in the case where differential equations possess these properties were considered in [1–3]. It should be noted that the inverse problem of the preservation of the indicated properties of solutions of differential equations in the case where the corresponding difference equations possess these properties is no less important. In [4], this problem was considered for the properties of stability and oscillation. In the present paper, we investigate this problem for the properties of boundedness and periodicity of solutions. We also study the problem of the convergence of periodic and bounded solutions of difference equations to the corresponding periodic and bounded solutions of differential equations.

2. Statement of the Problem

Consider the system of differential equations

$$\frac{dx}{dt} = X(t, x) \quad (1)$$

and the corresponding system of difference equations

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$$x_{k+1}^h = x_k^h + hX(t_0 + kh, x_k^h) \quad (2)$$

($k \in \mathbb{Z}$ and $h > 0$ is the step of the difference equation),

$$x_k^h = x_k^h(t_0 + kh),$$

$$x^h(t_0) = x_0^h.$$

The vector function $X(t, x)$ is defined for $t \in \mathbb{R}$, $x \in D$ (D is a certain domain in the space $\mathbb{R}^n$), and $t_0 \in \mathbb{R}$.

The aim of the present paper is to study the relationship between bounded and periodic solutions of systems (1) and (2).

3. Auxiliary Statements

To solve the problem posed above, we present several auxiliary statements. In what follows, we assume that the function $X(t, x)$ in the domain $\mathbb{R} \times D$ is continuously differentiable and bounded together with its partial derivatives, so that

$$|X(t, x)| + \left| \frac{\partial X}{\partial t} \right| + \left\| \frac{\partial X}{\partial x} \right\| \leq C.$$

Then the following lemma on the uniform estimate for the closeness of solutions of systems (1) and (2) is true:

Lemma 1. *Let $x(t)$ and x_k^h be solutions of the Cauchy problems defined on the segment $[t_0, t_0 + T]$ for systems (1) and (2) and such that $x(t_0) = x_0^h = x_0$, $x_0 \in D$. Then the following inequality is true:*

$$|x(t_0 + kh) - x_k^h| \leq he^{CT} [1 + NT] \quad \text{for } kh \leq T,$$

where $N = C + C^2$.

Proof. The validity of Lemma 1 follows directly from relation (12) in [5, p. 384].

Lemma 2. *Under the conditions formulated above, the solution x_k^h of system (2) depends continuously on the initial data up to the time of its exit from the domain D .*

Proof. Solutions of system (2) can be represented in the form

$$x_k^h = x_0^h + h \sum_{p=0}^{k-1} X(t_p, x_p^h),$$

$$y_k^h = y_0^h + h \sum_{p=0}^{k-1} X(t_p, y_p^h).$$

We have

$$|x_k^h - y_k^h| \leq |x_0^h - y_0^h| + h \sum_{p=0}^{k-1} |X(t_p, x_p^h) - X(t_p, y_p^h)| \leq |x_0^h - y_0^h| + h \sum_{p=0}^{k-1} C |x_p^h - y_p^h|.$$

This and the results of [1] yield

$$|x_k^h - y_k^h| \leq |x_0^h - y_0^h| (1 + Ch)^k,$$

which proves the lemma.

Lemma 3. *If the solution x_k^h of system (2) is uniformly asymptotically stable with respect to $k_0 \in \mathbb{Z}$, then it is also uniformly asymptotically stable with respect to x_0 .*

Proof. Since the solution x_k^h of system (2) is uniformly asymptotically stable, for any $\eta > 0$ there exists $\delta = \delta(\eta) > 0$ such that, for any other solution y_k^h of system (2), the inequality

$$|x_{k_0}^h - y_{k_0}^h| < \delta$$

yields

$$|x_k^h - y_k^h| < \frac{\eta}{2} \quad \text{for } k \geq k_0$$

and

$$\lim_{k \rightarrow \infty} |x_k^h - y_k^h| = 0. \quad (3)$$

Let $U_\delta(k_0)$ denote the δ -neighborhood of the point $x_{k_0}^h(x_0)$. We show that the limit relation (3) is uniform with respect to $x_0 \in U_\delta(k_0)$. Assume that this is not true. Then there exists $\mu > 0$ such that, in $U_\delta(0)$, one can find a convergent sequence of points $x^{(n)}$ and a sequence of numbers k_n such that

$$\begin{aligned} |x^{(n)} - x_0^h| &< \delta, \\ |x_{k_n}^h(x^{(n)}) - x_{k_n}^h| &\geq \mu, \quad k_n \rightarrow \infty, \end{aligned} \quad (4)$$

where $x_k^h(x^{(n)})$ is a solution of system (2) such that $x_0^h(x^{(n)}) = x^{(n)}$. Let

$$\lim_{n \rightarrow \infty} x^{(n)} = x^0.$$

Then

$$\lim_{k \rightarrow \infty} |x_k^h(x^0) - x_k^h| = 0,$$

and one can indicate $N > 0$ such that the following inequality is true for all $k \geq N + k_0$:

$$|x_k^h(x^0) - x_k^h| < \frac{\sigma(\mu)}{2}, \quad (5)$$

where $\sigma(\mu)$ is a constant that guarantees the inclusion of the solutions $x_k^h(x)$ ($x_0^h(x) = x$) of system (2) originating in the $U_{\delta(\mu)}(0)$ -neighborhood of the point $x_0^h(0)$ into the $(\mu/2)$ -neighborhood of the solution x_k^h for all $k \geq 0$.

By virtue of Lemma 1, there exists $N_1 > 0$ such that

$$|x_k^h(x^{(n)}) - x_k^h(x^0)| < \frac{\sigma(\mu)}{2}, \quad k \in [k_0, k_0 + N], \quad n > N_1 \quad (6)$$

(we can assume that $k_n \geq N + k_0$ for $n > N_1$).

Using inequalities (5) and (6), we get

$$|x_k^h(k_0 + N, x^{(n)}) - x_k^h(k_0 + N)| \leq \sigma(\mu).$$

Therefore,

$$|x_k^h(x^{(n)}) - x_k^h| < \frac{\mu}{2}, \quad k \geq k_0 + N.$$

Thus, for $k = k_n$ we obtain

$$|x_{k_n}^h(x^{(n)}) - x_{k_n}^h| < \frac{\mu}{2},$$

which contradicts inequality (4).

The lemma is proved.

4. Main Results

First, we consider the correspondence between bounded solutions of systems (1) and (2). The theorem below guarantees the existence of a bounded solution of system (2), provided that system (1) has a bounded solution.

Theorem 1. *If system (1) has a bounded solution $x(t)$ uniformly asymptotically stable with respect to $t_0 \in \mathbb{R}$, defined on $\mathbb{R}$, and belonging to the domain D together with a certain neighborhood of it, then there exists h_0 such that, for $h \leq h_0$, system (2) also has a bounded solution, and, furthermore,*

$$\sup_{k \in \mathbb{Z}} |x(kh) - x_k^h| \rightarrow 0, \quad h \rightarrow 0. \quad (7)$$

Proof. By the conditions of the theorem, $x(t)$ is a bounded solution of system (1), i.e., it satisfies the inequality

$$|x(t)| \leq A \quad \forall t.$$

By virtue of the uniform asymptotic stability of $x(t)$, for any $\varepsilon > 0$ and t_0 there exist $\delta > 0$ and T such that if $y(t)$ is a solution of system (1) for which the condition

$$|x(t_0) - y(t_0)| \leq \delta$$

is satisfied, then

$$|x(t) - y(t)| < \varepsilon \quad \text{for } t \geq t_0, \quad (8)$$

$$|x(t) - y(t)| \leq \frac{\delta}{2} \quad \text{for } t \geq t_0 + T, \quad (9)$$

and, furthermore, δ and T are independent of t_0 .

Let x_m^h be a solution of system (2) that, at the initial point t_0 , coincides with the solution $y(t)$, i.e.,

$$x_0^h = y(t_0). \quad (10)$$

Then, according to Lemma 1, the following inequality is true:

$$|y(t_0 + mh) - x_m^h| \leq he^{C(T+1)}[1 + N(T+1)] \quad \text{for } m \leq k_0,$$

where k_0 is such that $T \leq k_0 h \leq T + 1$. By virtue of stability, $y(t)$ for $t \geq t_0$ belongs to D together with a certain neighborhood.

We choose h so that

$$he^{C(T+1)}[1 + N(T+1)] \leq \frac{\delta}{2}. \quad (11)$$

Hence,

$$|y(mh) - x_m^h| \leq \frac{\delta}{2}, \quad m \leq k_0. \quad (12)$$

Further, we consider a segment of length $k_0 h$.

By virtue of condition (10), we have

$$|x_0^h - x(t_0)| \leq \delta.$$

It follows from (9) that

$$|x(t) - y(t)| \leq \frac{\delta}{2}, \quad t \geq k_0 h.$$

Therefore,

$$|x(mh) - x_m^h| \leq |x(mh) - y(mh)| + |y(mh) - x_m^h| < 2\varepsilon, \quad m \leq k_0,$$

and

$$|x(k_0h) - x_{k_0}^h| \leq \delta.$$

We now consider a segment of length k_0h — $[k_0h, 2k_0h]$ and choose the solution $\tilde{y}(t)$ of system (1) that starts from the point k_0h and coincides at this point with the solution x_m^h :

$$x_{k_0}^h = \tilde{y}(k_0h).$$

Using (13), we obtain

$$|x(k_0h) - \tilde{y}(k_0h)| \leq \delta.$$

Inequalities (8) and (9) yield

$$|x(t) - \tilde{y}(t)| < \varepsilon, \quad t \geq k_0h,$$

$$|x(t) - \tilde{y}(t)| \leq \frac{\delta}{2}, \quad t \geq k_0h + T.$$

Therefore, the inequalities

$$|x(mh) - x_m^h| < 2\varepsilon, \quad m \in [k_0, 2k_0]$$

and

$$|x(2k_0h) - x_{2k_0}^h| \leq \delta$$

are true.

Continuing the process of the construction of the solution x_m^h of system (2), we verify that x_m^h always remains in the 2ε -neighborhood of a bounded solution $x(t)$ of system (1), or in its δ -neighborhood for points of the form pk_0h .

Thus, x_m^h belongs to the 2ε -neighborhood of a bounded solution, and, hence, x_m^h is bounded for $mh \geq t_0$.

Therefore, for an arbitrary $t_0 \in \mathbb{R}$, we have constructed a bounded solution x_m^h of Eq. (2) for $mh \geq t_0$, where h is independent of t_0 . Let us construct a solution x_m^h of Eq. (2) bounded on the entire axis. By virtue of the results presented above, all solutions of the difference equation that start from the δ -neighborhood of the solution $x(t)$ do not leave its 2ε -neighborhood, and after k_0 steps they return to its δ -neighborhood. Therefore, the bounded solutions x_m^h of the difference equation that start at time $-k_0$ from the δ -neighborhood of the point $x(-k_0h)$ return to the δ -neighborhood of the point $x(0)$ for $m = 0$.

By analogy, one can show that all bounded solutions of system (2) that start at $-k_0n$ from the δ -neighborhood of the point $x(-k_0nh)$ do not leave the 2ε -neighborhood of the solution $x(t)$ and, for $m = -k_0(n-1)$, they hit the δ -neighborhood of the point $x(-k_0(n-1)h)$.

Let $\mathbb{S}_n$ denote the range of values of solutions of system (2) at the point $m = 0$ that belong to the δ -neighborhood of the point $x(-k_0nh)$ for $m = -k_0n$. This implies that this set is not empty for an arbitrary natural n and $\mathbb{S}_n \subset \mathbb{S}_{n-1}$. By construction, the sets $\mathbb{S}_n$ consist of the images of solutions of Eq. (2) that start from the points $m = -k_0n$. By virtue of Lemma 2, the mapping that generates $\mathbb{S}_n$ is continuous, and, hence, $\mathbb{S}_n$ are closed (as the images of closed sets under a continuous mapping).

Let z_0 be a point common for all $\mathbb{S}_n$. Now consider a solution $x_m^h(z_0)$ of the difference equation that, for $m = 0$, starts from the point z_0 . This solution is, by construction, extendable to the left, and, at the points $-k_0n$, it belongs to the δ -neighborhood of the point $x(-k_0nh)$ for any natural n ; here, $x(t)$ is the bounded solution of system (1) indicated in the theorem. Therefore, it is infinitely extendable to the left and belongs to the 2ε -neighborhood of the solution $x(t)$, and, hence, it is bounded. Its extendability to the right and, hence, its boundedness are obvious. This proves the first part of the theorem.

It follows from the proof of the theorem that inequality (11) holds for all $h_1 < h$ chosen according to this inequality. Therefore, for these h_1 , system (2) has a bounded solution that does not leave the 2ε -neighborhood of the solution $x(t)$. Since ε is arbitrary, we conclude that relation (7) is true.

The theorem is proved.

We now consider the problem of the correspondence between periodic solutions of Eqs. (1) and (2). Let a function $X(t, x)$ be periodic in t with period ω , i.e., $X(t + \omega, x) = X(t, x)$. We take the step $h = \omega/n$, where n is a natural number.

Theorem 2. *If, for a sufficiently small step h ($n \geq N_0$), uniformly in k_0 and h system (2) has an asymptotically stable periodic solution x_k^h that belongs to D together with a certain ρ -neighborhood, then system (1) also has a periodic solution with a period that is a multiple of ω .*

Proof. By the conditions of the theorem, we have

$$\forall \varepsilon > 0 \quad \left(\varepsilon < \frac{\rho}{2} \right) \quad \exists \delta > 0 \quad (\delta < \varepsilon) \quad \exists n_0(\delta) \in \mathbb{N}.$$

If $|x_{k_0}^h - y_{k_0}^h| < \delta$, then

$$|x_k^h - y_k^h| < \frac{\varepsilon}{2} \quad \text{for } k \geq k_0 \quad (14)$$

and

$$|x_k^h - y_k^h| < \frac{\delta}{2} \quad \text{for } k \geq n_0 + k_0. \quad (15)$$

Moreover, δ is independent of k_0 and h .

For given $\delta > 0$ and n_0 , we choose the step

$$h_0 = h_0(\delta, n_0) = \frac{\omega}{m_0}$$

such that if $y_k^{h_0}$ is a solution of the difference equation and $\varphi(t)$ is a solution of the differential equation such that $\varphi(kh_0) = y_k^{h_0}$, then

$$|\varphi(ih_0) - y_i^{h_0}| < \frac{\delta}{2}, \quad i \in [k, k + n_0]. \quad (16)$$

By virtue of Lemma 1, this choice of h_0 is possible. By the conditions of the theorem, for this h_0 system (2) has a periodic asymptotically stable solution $x_k^{h_0}$ of period $p(h_0)$.

Now consider the δ -neighborhood $U_\delta(x_0^{h_0})$ of the point $x_0^{h_0}$. Let y_0 be an arbitrary point from $U_\delta(x_0^{h_0})$, let $\varphi(t, y_0)$ be a solution of system (1) such that $\varphi(0, y_0) = y_0$, and let $y_k^{h_0}$ be a solution of system (2) such that $y_0^{h_0} = y_0$.

Using relations (14) and (15), we get

$$|y_k^{h_0} - x_k^{h_0}| < \frac{\varepsilon}{2}, \quad k \in [0, n_0],$$

$$|y_{n_0}^{h_0} - x_{n_0}^{h_0}| < \frac{\delta}{2},$$

and

$$|x_k^{h_0} - \varphi(kh_0, y_0)| < \varepsilon, \quad k \in [0, n_0],$$

$$|x_{n_0}^{h_0} - \varphi(n_0h_0, y_0)| < \delta.$$

Thus, the solution $\varphi(t)$ of system (1) that starts from the δ -neighborhood of $x_0^{h_0}$, without leaving the ε -neighborhood of the solution $x_k^{h_0}$ of system (2) at the nodes kh_0 ($k \in [0, n_0]$), hits its δ -neighborhood again at time n_0h_0 , provided that the solution $\varphi(t)$ is defined on the segment $[0, n_0h_0]$. Let us show that this can be realized by choosing a sufficiently small step h .

We choose $\varepsilon > 0$ so that the points from the ε -neighborhood of the periodic solution x_k^h of system (2) belong to the domain D together with their $(\rho/2)$ -neighborhood. By virtue of the conditions of theorem, this choice is possible. Therefore, the solutions of system (1) that start from this $(\rho/2)$ -neighborhood can be extended to the left and to the right onto an interval whose length is at least $\rho/(2C)$. We now choose a step h_0 so that inequalities (16) are true and $h_0 < \rho/(2C)$. As a result, we get $h_0 = \omega/m_0$, where m_0 is a certain natural number.

Then a solutions $\varphi(t)$ that starts from the δ -neighborhood of $x_0^{h_0}$ can be extended to the segment $[0, \rho/(2C)]$, and the following inequality holds at the point $t = h_0$:

$$|\varphi(h_0) - y_1^{h_0}| < \frac{\delta}{2},$$

where $y_k^{h_0}$ is the solution of system (2) indicated above. Thus, the point $\varphi(h_0)$ belongs to the domain D together with its $(\rho/2)$ -neighborhood. Hence, the solution $\varphi(t)$ can be extended to the point $2h_0$, and $\varphi(2h_0)$ also belongs to D together with its $(\rho/2)$ -neighborhood. Continuing this process, we establish that $\varphi(t)$ is defined on the segment $[0, n_0 h_0]$.

Let $\hat{y}_k^h$ denote a solution of system (2) whose initial data at time n_0 coincide with the initial data of the solution $\varphi(t)$, i.e., $\varphi\left(\frac{n_0 \omega}{m_0}\right) = \hat{y}_{n_0}^{h_0}$.

By analogy with the arguments presented above, we get

$$|\hat{y}_k^{h_0} - x_k^{h_0}| < \frac{\varepsilon}{2}, \quad k \in [n_0, 2n_0],$$

$$|\hat{y}_{2n_0}^{h_0} - x_{2n_0}^{h_0}| < \frac{\delta}{2},$$

and

$$|x_k^{h_0} - \varphi(kh_0)| < \varepsilon, \quad k \in [n_0, 2n_0],$$

$$|x_{2n_0}^{h_0} - \varphi(2n_0 h_0)| < \delta.$$

Continuing this process, at the M th step, where M is the least common multiple of the numbers m_0 , n_0 , and p , we get

$$|x_M^{h_0} - \varphi(Mh_0)| < \delta.$$

Moreover, $Mh_0 := M^0 = r\omega$, and it is a multiple of ω . Hence,

$$|x_M^{h_0} - \varphi(r\omega)| < \delta.$$

Since $x_M^{h_0} = x_0^{h_0}$, the mapping $\pi: y_0 \rightarrow \varphi(r\omega, y_0)$ maps a ball of radius δ into itself. Therefore, there exists a fixed point $y_1 \in U_\delta(x_0^{h_0})$ of the mapping such that $\varphi(r\omega, y_1) = y_1$. This means that a solution of Eq. (1) with initial condition $\varphi(0) = y_1$ is periodic with period $r\omega$.

The theorem is proved.

The result presented below establishes conditions for the convergence of periodic solutions of system (2) to a periodic solution of system (1).

Theorem 3. *If, under the conditions of Theorem 2, system (1) has a unique periodic solution $\varphi(t)$ of period $s\omega$, where s is an integer, then the following relation is true:*

$$\lim_{n \rightarrow \infty} \sup_{m \leq n} \left| x_m^h \left(\frac{ms\omega}{n} \right) - \varphi \left(\frac{ms\omega}{n} \right) \right| = 0, \quad (17)$$

where x_m^h is a periodic solution of system (2) and $h = s\omega/n$ is the step.

Proof. To prove relation (17), it suffices to show that, for any $\varepsilon > 0$, there exists h_0 (there exists n_0 : $h_0 = s\omega/n_0$) such that if $h < h_0$ ($n > n_0$), then

$$\left| \varphi \left(\frac{s\omega}{n} k \right) - x_k^h \left(\frac{s\omega}{n} k \right) \right| < \varepsilon, \quad k \leq n. \quad (18)$$

By virtue of Theorem 2, system (2) has a periodic solution for all sufficiently small h , and, furthermore, for $\varepsilon > 0$ indicated above, there exist $\delta > 0$ and $h_0 = h_0(\delta)$ such that, for $h < h_0$, a certain periodic solution $\varphi(t)$ of system (1) starts from the δ -neighborhood of x_0^h , where x_k^h is a periodic solution of system (2), and, furthermore, this solution differs at the nodes from the corresponding periodic solution x_k^h by at most ε for all $h < h_0$. By virtue of the uniqueness of the periodic solution $\varphi(t)$ of system (1), inequality (18) is true, which completes the proof of the theorem.

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