

A HYBRID METHOD OF STUDYING THE STRESS INTENSITY FACTORS AT THE TIPS OF CRACKS IN ANISOTROPIC PLATES

V. N. Maksimenko and N. M. Yungerman

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We propose an efficient new method of numerical analysis of the residual strength of finite plates weakened by cracks. The basis of the method is an alternating Schwartz procedure that makes it possible to achieve a successful combination of the algorithmicity and indifference to boundary conditions of the finite-element method and the application of the method of integral equations for studying singular stress fields. The numerical implementation of the scheme and the algorithm for determining the stress intensity factors at the tips of cracks is given as a software package written in FORTRAN. The efficiency of application of the computational methodology is illustrated by examples.

3 Figures. 4 Tables. Bibliography: 6 titles.

Determination of the stress intensity factors at the tips of slits of arbitrary shape in finite anisotropic plates encounters certain difficulties even when such efficient numerical methods as the integral equation method and the finite-element method are applied. In the case of the finite-element method, despite its universality and algorithmic nature, simplicity of solution for regions of complicated shape, and indifference to boundary conditions, the difficulties are connected with the precise analysis of the singularity of the stress fields. The successful application of the method of integral equations in studying singular stress fields is complicated by the finite shapes of the body and the irregularities of the boundaries. The use of special finite elements or the solution of the additional integral equations on the boundary of the body does not completely free the methods from these defects, yet the cumbersomeness of the computations increases significantly.

The essence of the proposed approach, which preserves the advantages of the finite-element method and the method of integral equations consists of combining them using the alternating Schwartz method of [1]. Its basic idea and computations are simple, but to date it has found few applications to the solution of practical problems.

Consider an anisotropic plate occupying the region D with boundary C in the xy -plane. The plate is weakened by a curvilinear slit L . A system of external forces is applied to the boundary C :

$$F(t) = \sigma_n(t) + i\tau_n(t), \quad t \in C, \quad (1)$$

where σ_n and τ_n are the normal and tangential components of the vector $F(t)$. The edges of the slit L are free of load.

We now break the problem into two parts (Fig. 1) and construct two sequences of corresponding solutions. In the first of these we exclude the slit and, using the finite-element method, we obtain a sequence of solutions (A) for the undamaged plate (the region D_0) in the form of the forces

$$\{-p_j(t)\} = \{-[\sigma_n(t) + i\tau_n(t)]\} \quad (2)$$

on the edges of an imaginary slit L' corresponding to the sequence of loads $F_j(t)$, ($j = 0, 1, 2, \dots$). The second sequence of solutions (B) corresponds to an infinite plate (region D_∞) with a real crack L whose edges are loaded by forces $p_j(t)$. These solutions can be obtained in the form of potentials [2] using the method of integral equations

$$\Phi_\nu^{(j)}(z_\nu) = \frac{1}{2\pi i} \int_L \frac{\omega_\nu(\tau_\nu) d\tau_\nu}{\tau_\nu - z_\nu}, \quad (3)$$

Translated from *Teoreticheskaya i Prikladnaya Mekhanika*, No. 23, 1992, pp. 34–40.

Table 1

φ	0				$\pi/2$			
l/W	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
$k_1^{(0)}$	<u>1.057</u> 1.033	<u>1.133</u> 1.061	<u>1.205</u> 1.081	<u>1.271</u> 1.271	<u>1.057</u> 1.033	<u>1.133</u> 1.062	<u>1.205</u> 1.082	<u>1.271</u> 1.271
$k_1^{(1)}$	<u>1.115</u> 1.075	<u>1.299</u> 1.144	<u>1.460</u> 1.146	<u>1.508</u> 1.508	<u>1.118</u> 1.078	<u>1.300</u> 1.175	<u>1.480</u> 1.251	<u>1.582</u> 1.582
$k_1^{(2)}$	<u>1.118</u> 1.077	<u>1.319</u> 1.153	<u>1.501</u> 1.155	<u>1.550</u> 1.550	<u>1.121</u> 1.080	<u>1.322</u> 1.189	<u>1.541</u> 1.282	<u>1.651</u> 1.651
$k_1^{(3)}$		<u>1.321</u> 1.154	<u>1.508</u> 1.156	<u>1.557</u> 1.557		<u>1.325</u> 1.191	<u>1.554</u> 1.287	<u>1.666</u> 1.666
$k_1^{(4)}$			<u>1.509</u> 1.157	<u>1.558</u> 1.558		<u>1.325</u> 1.191	<u>1.556</u> 1.288	<u>1.667</u> 1.667
$k_1^{(5)}$						<u>1.326</u> 1.191	<u>1.557</u> 1.288	
$k_1^{(6)}$							<u>1.557</u> 1.289	
k_1^*	<u>1.118</u> 1.077	<u>1.321</u> 1.154	<u>1.509</u> 1.157	<u>1.558</u> 1.558	<u>1.121</u> 1.080	<u>1.326</u> 1.191	<u>1.557</u> 1.289	<u>1.667</u> 1.667
k_1^*					<u>1.117</u> 1.075	<u>1.311</u> 1.151	<u>1.483</u> 1.213	<u>1.624</u> 1.624

Table 2

j	$\varphi = 0$			$\varphi = \pi/2$		
	l/W					
	0.2	0.4	0.6	0.2	0.4	0.6
0	1.041	1.041	1.041	1.041	1.041	1.041
1	1.243	1.488	1.551	1.269	1.479	1.624
2	1.288	1.722	1.965	1.328	1.735	2.064
3	1.298	1.857	2.305	1.343	1.886	2.419
4	1.301	1.933	2.585	1.347	1.976	2.710
5		1.975	2.814	1.348	2.029	2.949
6		1.999	3.003		2.060	3.144
7		2.012	3.158		2.078	3.304
8		2.019	3.286		2.089	3.435
9		2.023	3.391		2.096	3.543
10		2.025	3.477		2.100	3.631
11		2.027	3.547		2.102	3.703
12			3.606		2.103	3.762
13			3.653		2.104	3.811
14			3.693		2.105	3.850
23			3.843			4.001
30			3.866			4.023
37			3.872			4.029
k_1	1.301	2.027	3.872	1.348	2.105	4.029
k_1^*	1.317	2.042	3.859			

The efficiency of this method is illustrated by the example of the computation of the stress intensity factors (Tables 1-4) for interior and boundary cracks under distention (Fig. 2) and bending in its plane (Fig. 3), where in the finite-element method we used as finite elements the isoparametric rectangular anisotropic squares described in [5]. The discrete finite-element model of the region being computed forms

Table 3

j	Material 1								Material 2			
	$\varphi = 0$				$\varphi = \pi/2$				$\varphi = 0$		$\varphi = \pi/2$	
	l/V											
	0.1	0.2	0.3	0.4	0.1	0.2	0.3	0.4	0.1	0.2	0.1	0.2
0	0.897	0.772	0.645	0.517	0.899	0.773	0.646	0.517	0.920	0.793	0.926	0.800
1	0.962	0.944	0.898	0.816	0.974	0.963	0.910	0.813	0.977	0.952	0.990	0.972
2	0.967	0.978	0.988	0.983	0.980	1.012	1.026	0.990	0.980	0.984	0.994	1.010
3		0.987	1.037	1.077	0.981	1.024	1.077	1.095		0.990		1.018
4		0.988	1.052	1.128		1.028	1.099	1.157		0.991		1.020
5		0.989	1.058	1.157		1.029	1.108	1.193				
6			1.060	1.174		1.112	1.215					
7			1.061	1.182		1.114	1.228					
8			1.061	1.187		1.115	1.235					
9			1.062	1.190		1.115	1.240					
10				1.192		1.115	1.242					
11				1.193		1.116	1.244					
12			1.193					1.245				
k ₁	0.967	0.989	1.062	1.194	0.981	1.029	1.116	1.245	0.980	0.991	0.994	1.020
k ₁ [*]					0.991	1.013	1.083	1.211				

Table 4

N	5	9	16	23
N_2	3	5	10	15
$k_1^{(0)}$	$\frac{1.057}{1.033}$	$\frac{1.057}{1.033}$	$\frac{1.057}{1.033}$	$\frac{1.057}{1.033}$
$k_1^{(1)}$	$\frac{1.104}{1.066}$	$\frac{1.112}{1.072}$	$\frac{1.115}{1.075}$	$\frac{1.118}{1.078}$
$k_1^{(2)}$	$\frac{1.106}{1.067}$	$\frac{1.115}{1.074}$	$\frac{1.118}{1.077}$	$\frac{1.121}{1.080}$

a rectangular grid of nodes along the coordinate axes (Fig. 2). The recursive process terminates when $|K_1^{(j+1)} - K_1^{(j)}| < 10^{-4}$.

If L is a rectilinear crack, then the solutions $\Phi_\nu^{(j)}(z_\nu)$ of the sequence (B) are given immediately by quadratures in terms of the forces $p_j(t)$ on the edges L (the integral equations (4) can be solved in closed form and the need to solve the integral equations numerically on L disappears).

The structural clarity of the alternating Schwartz procedure assured the good algorithmicity of the proposed hybrid method. The algorithm is implemented as a software package. The block architecture of the package made possible some further extensions and modernization. The sequence of solutions (A) is executed by the finite-element method in the classical form of the method of displacements.

Table 1 gives the values of the normalized stress intensity factors $k = K_1/\sigma\sqrt{\tau}$ at the left- and right-hand tips of the interior crack (above and below the bar respectively, cf. Fig. 2, where $h/W = 0.2$) under distention of the plate for the orthotropic Material 1 ($E_1 = 170.6$ GPa, $E_2 = 55.2$ GPa, $G_{12} = 4.83$ GPa, $\nu_1 = 0.11$) for various l/W , where $k^{(j)}$, k , k_1 , and k_1^* are the normalized stress intensity factors at the j th recursion, at the last recursion, and the data of [6] respectively for an infinite strip. For the case of a boundary crack (Fig. 2 with $h = 0$) and the same material the values of the normalized stress intensity factors are given in Table 2. Table 3 contains the values of the normalized stress intensity factors of a plate

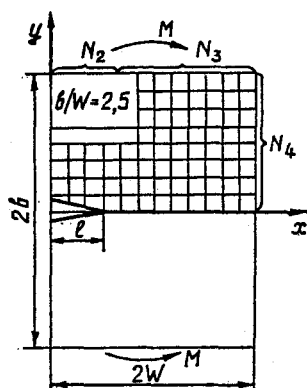


Fig. 2

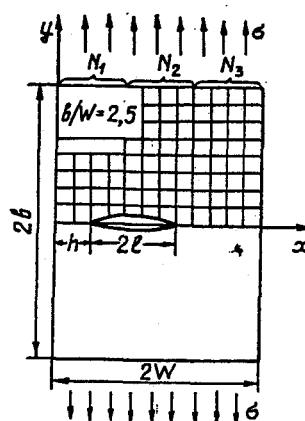


Fig. 3

with a boundary crack under bending (Fig. 2 with $\sigma = 3M/(2W)$) for Material 1, and also Material 2 ($E_1 = 223.4$ GPa, $E_2 = 24.13$ GPa, $G_{12} = 8.48$ GPa, $\nu_1 = 0.23$). The results were obtained using the special representations (6) with $N_4 = 5$, $N_2 = 15$, $N_1 = N_3 = 4$ for an interior asymmetric crack and $N_3 = 8$ for a boundary crack. When $l/W = 0.8$ (a symmetric crack), the solution (3) was used. In both cases the number of Chebyshev nodes on the contour L was set equal to 10.

Table 4 gives the results of a study of convergence of the recursion process for Material 1 as a function of the number of finite elements $N = \sum_{j=0}^3 N_j$ over the width of the plate (Fig. 2). Here it was assumed that $l/W = 0.02$, $h/W = 0.2$, and $\varphi = \pi/2$.

The results obtained admit the following conclusions.

Increasing the degree of anisotropy, letting the tips of an internal crack approach the rim of the plate, and increasing the length of the boundary crack worsen the convergence of the procedure. Using the potentials (6) significantly improves the convergence. For example, for an interior crack (Fig. 1, Material 1, $\varphi = \pi/2$, $l/W = 0.4$, $h/W = 0.2$) the use of (3) gives $k_1^{(5)} = 1.102$ for the left (resp. right) tip of the crack; the representations (6) yield $k_1^{(5)} = 1.326$ (resp. 1.191). Kaya and Endoran [6] obtained $k_1 = 1.311$ (resp. 1.151) for this case. For large values of h/W the advantages of the solutions (6) disappear and it becomes more reasonable to use the closed-form solutions (in quadratures) (3).

The finite-element partition has only weak influence on the computed values of $k^{(j)}$. Even with $N_1 = N_3 = 1$ and $N_2 = 3$ the values of $k_1^{(2)}$ differ from those given in [6] by less than 1%.

The algorithm of the proposed method can be easily adapted to various types of programs for implementing the solution of the sequences (A) and (B). An acceptable precision for engineering computations is reached in the early recursions. The algorithm makes economical use of the machine resources of memory and computing time by saving the constants of the matrices of the system of linear algebraic equations that give the solutions of the sequences (A) and (B). This in combination with the possibility of achieving an acceptable precision in the result when using a finite-element grid of small size makes it possible to recommend the method just discussed for use on personal computers.

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