

ANALYTIC METHODS OF CALCULATION OF THE CONTACT INTERACTION OF THIN-WALLED STRUCTURAL ELEMENTS (REVIEW)

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We describe main models and methods for the calculation of contact rigidity and wear resistance of friction couples using thin-walled structural elements such as beams, plates, or shells. We analyze certain specific features of their contact interaction on the basis of theories describing stress-strain state. We also consider mathematical models of the wear of junctions that take into account the frictional heating of contacting surfaces and anisotropy of their properties. A series of theoretical calculations is corroborated experimentally.

Modern machines and equipment must work reliably under operating conditions close to critical. As a rule, one of the main causes of their failure is the wear of triboconjugations, where parts are subject to contact interaction with one another or with abrasive components of the environment. This leads to changes in their geometry and physicochemical and chemical properties of materials of which these parts are made and produces a negative effect on the durability and reliability of the structures as a whole.

Thin-walled structural elements are widely used due to their versatility and the fact that their production does not require a large amount of metal. For example, cylindrical thin-walled parts are used in kinematic links of machines, instruments, and devices as guide and load-bearing elements, which assumes the mutual displacement of coupled surfaces and, hence, their wear [1, 2].

This explains the necessity of comprehensive investigation of the contact interaction of thin-walled structural elements with regard for the phenomena that accompany this process. Since there are no general works on this subject, we analyzed the available literature covering the most important achievements in the field of contact interaction with regard for the wear of not only homogeneous thin-walled elements but also of inhomogeneous ones (with coatings or surface layers modified by other methods). We focused our attention on the specific features of existing approaches to the construction of the corresponding theories, models, and computation methods.

As is known, the law of wear serves as a basis for the construction of theories and methods for the calculation of wear of friction assemblies. In general, the process of wear is complex [3–5] and includes complex physicochemical, thermal, and mechanical phenomena induced by the formation and destruction of a frictional contact (variation of its geometric characteristics, kinematics of motion, changes in the structure and composition of the surface layers of the material, and the formation of chemical compounds on the contact surfaces), lubrication, etc. [6–9]. Hence, the law of wear is a generalized criterial equation that relates the wear intensity to physicochemical and thermophysical characteristics of the materials under consideration, their strength, and the roughness of contact surfaces [5, 8, 10], namely,

$$J = K_0 \left(\frac{p}{E_* \epsilon^\tau} \right)^x \left(\frac{HB}{\sigma_c} \right)^y \left(\frac{RT}{U_0} \right)^z \left(\frac{f^\tau p V h_k}{\lambda_1 + \lambda_2} \right)^m m_0^n, \quad (1)$$

where K_0 is a coefficient, p is the contact pressure, E_* is the reduced modulus of elasticity,

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$$\frac{1}{E_*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2},$$

ν_1 , ν_2 and E_1 , E_2 are the Poisson's ratios and Young's moduli, respectively, of the contacting bodies, ε , HB , and σ_c are the relative elongation after rupture, hardness, and compressive strength of the less strong material, τ is the parameter of frictional fatigue, RT is the energy of thermal motion, the fluctuations of which are responsible for the breakage of interatomic chemical bonds weakened due to contact interaction, U_0 is the activation energy of breakage, f is the friction coefficient, V is the velocity of mutual slippage, h_k is the distance between the bodies, λ_1 and λ_2 are thermal conductivities, and m_0 is the form factor of the surface.

The final form of relation (1) depends substantially on what factors are taken into account. No matter what equations are used for the description of the wear-contact problem (interrelated equations of the theory of elasticity, heat conduction, diffusion, fluid dynamics, etc.), regardless of whether the dynamic terms are taken into account or not, the parameter of time t is always present in the law of wear (1).

Equation (1) includes many parameters of the process under study, but an analysis of the experimental data available demonstrates that the most important parameters are the distance between the surfaces, contact stresses and strains (especially in the surface layers), frictional heating, temperature distribution, and, finally, the roughness of the surfaces. Designers are interested in numerical models of wear resistance in which the law of wear (1) includes the majority of the main factors of the problem characteristic of typical friction and wear assemblies (gears, rolling and sleeve bearings, cam mechanisms, sleeve bearings, packings of shafts, etc.); at the same time, Eq. (1) should take as simple form as possible. Most often, the power dependence of the wear intensity (rate) on pressure [1, 11–18] is used, namely,

$$\frac{\partial w_*}{\partial t} = fVK_0p^\alpha(x, y, t) = K_w p^\alpha(x, y, t). \quad (2)$$

Here, $0 \leq \alpha \leq 2$, w_* is the change in the sizes of the contacting bodies due to wear in the direction normal to the contact domain, t is the wear time, and (x, y) is a point belonging to this domain. Formula (2) is characteristic of fatigue wear. In engineering practice, as a rule, one uses a linear dependence of the wear rate on the contact pressure, namely,

$$\frac{\partial w_*}{\partial t} = K_w p(x, y, t). \quad (3)$$

If $K_w = K_w(t)$, then one can take the effect of aging into account (the hereditary model of wear [1, 19]).

The first investigations in the field of tribology date back to the works of Leonardo da Vinci, who established that the friction force is proportional to a normal pressing force and studied the resistance of individual bodies to motion, antifriction materials, and lubricants. Similar problems were studied by G. Amontons, G. Galilei, L. Euler, Ch. Coulomb, M. V. Lomonosov, I. Newton, etc. [20–38].

The calculation of wear resistance, the prediction of service life, the optimization of triboconjugations in the science of machines and machine building, and a wide spectrum of investigations in the field of physics, chemistry, and mechanics of friction processes are described in detail in [39–46]. The contact problem forms a basis for the calculation of the wear of parts. In determining the character and intensity of wear, contact stresses and strains serve as the main parameters.

The first attempts to solve the problem of contact of elastic bodies (the contact domain is fairly small compared with the radii of curvature of the bodies under consideration, these bodies are perfectly elastic, and friction

in the contact domain is absent) were made by H. Hertz; the corresponding results were published in his classical work "On the contact of elastic bodies" [47]. Later, many scientists obtained fundamental results in the field of contact problems of elasticity (see surveys in [48–58]).

In the last decades, especially in connection with the development of computers, numerical methods (the variational-difference method, the finite-element method, the method of boundary integral equations, etc.) have been extensively used for the solution of contact problems [59–66]. This enables one to consider contact interaction in assemblies of machines and mechanisms for which it is difficult to apply classical solutions.

Since thin-walled structural elements, new composite materials, and bodies with coatings, interphase interlayers, patches, and stiffening ribs are widely used in general and special machine building, contact problems for these structures draw much attention from researchers [33, 35, 39, 40]. On the one hand, this class of contact problems is closely related to classical contact problems in the mechanics of deformable solids and, on the other hand, it is characterized by certain features of load transfer in the course of contact of thin-walled structures.

The contact of thin-walled structural elements (bars, plates, and shells) is a specific problem as compared with classical problems in the theory of elasticity due to the substantial difference between the structure of contact stresses and the specific character of the formulation of these problems because the basic Hertz hypothesis about the smallness (locality) of contact domains cannot be used here. In the case of contact of thin-walled structural elements, one of the sizes of the body under consideration (thickness) is small compared with the other sizes and is commensurable with the contact domain. There is also another difference, namely, for thin-walled elements it is necessary to take into account the conditions of fastening.

For classical contact problems, a numerical model that describes the stress-strain state of a body was developed earlier. An analysis of the contact interaction of thin-walled elements shows that the choice of a model of a thin-walled element affects the final result. For example, if the contact pressure between a rigid die without angular points and a thin-walled element is determined by using the Kirchhoff–Love theory, then one should take into account concentrated forces in the solution obtained. It follows from a solution of the model problem of the pressure of a parabolic die on a gripped plate of finite length (plane problem) that, to obtain the size of the contact domain equal to the plate width, one should apply an infinite compressive force, which is unreal. However, solving the same problem by using a theory that takes into account lateral shear, one obtains a solution in the class of continuous functions (there are no concentrated forces), although the contact pressure is not equal to zero at the ends of the contact domain. Taking into account the effect of transverse pressing of the normal to the median surface of the plate, one can obtain a physically correct solution. Note that the distribution of contact stresses depends substantially on the theory used, but, in general, the stress-strain state of a thin-walled element determined under these contact loads differs insignificantly. The main achievements in the field of contact problems for thin-walled structural elements are summarized in [67–76].

The most widely used method for the solution of contact problems for thin-walled structural elements is the method of conjugation [77–79]. According to it, the system of key equations is solved separately in the contact domain and outside it, and the solutions obtained are conjugated along the boundary of the contact domain. This method is efficient for one-dimensional or axially symmetric problems, for which one can easily obtain an analytic solution in each domain.

The semiinverse method [67, 70, 78] is also extensively used. In this case, taking into account the statement of the problem, one chooses an expression for a certain stress in the form of an unknown continuous function and a system of concentrated forces whose values are unknown and which are given at certain points of the contact domain. Further, one determines a solution as in the case of a standard boundary-value problem with given loading. Unknown constants appearing in the expressions for contact stresses are calculated from the condition of contact. This method is efficient if the structure of a solution is known beforehand.

At present, the method of fundamental solutions [70] is extensively developed and used. It is based on the construction of influence functions, which, in fact, are solution of boundary-value problems corresponding to the action of a unit force applied at an arbitrary point of a thin-walled element. In this case, using the principle of

superposition and the condition of contact, one constructs a Fredholm integral equation for finding contact pressure. An analysis of this equation shows that, within the framework of the Kirchhoff–Love theory, we obtain an integral equation of the first kind, the solution of which is mathematically incorrect [80]. Therefore, one seeks this solution in the class of generalized functions. If a theory that takes into account lateral-shear deformation is used for modeling the stress-strain state of a thin-walled element, then one again arrives at a Fredholm integral equation of the first kind. Its kernel enables one to reduce the problem under study to the solution of an integral equation of the second kind and to construct a solution in the class of continuous functions for dies without angular points. For a plane die, it is necessary to find a nonzero solution of a homogeneous Fredholm integral equation of the second kind, which is connected with the problem of finding the eigenvalues of its kernel [70, 77, 79]. Taking into account lateral-shear deformation, we get a bounded solution, but it does not vanish at the boundary of the contact domain.

By virtue of the assumption on transverse pressing of the normal to the median surface of the thin-walled element, one obtains an inhomogeneous Fredholm integral equation of the second kind, the solution of which always exists for dies of arbitrary shape [81]. The unknown contact domain and rigid displacement of the die are determined from the conditions of equilibrium of the die and absence of contact pressure at the boundary of the contact domain [67–70, 75, 77].

The method of Volterra integral equations in contact problems for thin-walled structural elements was used in [75, 76, 82, 83]. The well-developed theory of these equations enables one to easily analyze the structure of a solution, depending on the chosen model.

As follows from the arguments presented above, the result of the solution of contact problems for thin-walled structural elements depends on the theory used for the description of the stress-strain state of an elastic body, whereas the quality of this theory, its adequacy, and, at the same time, its simplicity are determined by the procedure of its construction.

Earlier, the majority of numerical schemes for the wear of friction assemblies neglected the elasticity of materials or were based on the Hertz solutions, which imposed numerous substantial restrictions. Galin [84, 85] first applied achievements of the mechanics of contact interaction to the problem of wear. He posed a contact problem for an elastic body with regard for wear and proposed an approximate method for the solution of the corresponding integral equation, which depends on space coordinates and time; he also posed the problem of the wear of a Kirchhoff beam [84, 85].

Numerous works are devoted to the wear of sleeve bearings [1, 7, 24, 30–34, 36–38]. Some authors [5, 24, 29] assume that the contact domain is small compared with the sizes of the contacting bodies, whereas other researchers [1, 33, 34, 41] do not use this restriction. Numerous authors studied the wear of an antifriction element in a sleeve bearing, the kinetics of wear of an elastic circular bushing in contact with an elastic shaft for different laws of the process [36, 38, 19], contact problems with regard for properties of surfaces of solids, and the friction effects attributable to viscoelastic properties of materials. In the literature, one can find the results of investigations of the contact interaction of bodies in the case of internal contact of coaxial cylinders with close radii and faceted contours [1]. The wear of coatings of elastic bodies by a rigid indenter was considered in [1, 75, 86, 87], and analytic solutions of the problems of contact of two plane-parallel layers and two hollow circular cylinders (rings) were constructed in [25].

Another main factor in wear-contact problems (second to contact stresses and strains in importance) is temperature. In the course of operation of a friction couple, energy is spent on deformation, heat (a major part), the formation of defects and dislocations, etc. High temperature and heat flow decrease the strength of materials, accelerate chemical processes in the contact domain, change the structural and phase states of materials, lead to additional stresses, etc. Numerous investigations were devoted to studying the role of temperature, heat generation, and the relationship between mechanical, thermal, diffusion, and hydrodynamic fields. A fairly extensive survey of these works, especially those published in recent years, can be found in [88, 89].

To construct a mathematical model of heat generation caused by the action of friction forces, one should consider the thermoelastic problem of contact interaction of bodies with unknown intensities of heat sources in the contact domain. The conditions of heat generation due to friction were formulated in [90–96] as follows:

- at every point of the contact domain, the sum of rates of the heat flows directed into the bodies is equal to the specific work of friction forces;
- in the case of perfect thermal contact, the temperatures of the bodies are equal;
- if one takes into account the roughness of the surfaces, the presence of a lubricant, etc., i.e., the presence of a so-called third body [84, 85, 90, 91], then, instead of the condition of equality of temperatures, the conditions of imperfect thermal contact are used. In the latter case, it is customary to assume that the difference of heat fluxes is proportional to the difference of temperatures of the bodies; the corresponding coefficient of proportionality is called the coefficient of thermal conductance of the contact area (the inverse quantity is the thermal resistance). Pidstryhach [96] first formulated the conditions of imperfect thermal contact. Complex investigations of thermoelastic contact problems with regard for roughness, friction, wear, and frictional heat generation were carried out by Hrylits'kyi and his disciples [86, 87, 97–100]. Important results in the field of contact interaction of elastic bodies with regard for heat generation were obtained by Korovchinskii [91, 92]. The works [93, 94, 101] are devoted to the investigation of the quasistationary mode of heat generation in the contact domain of two elastic bodies.

The theoretical investigation of thermal and thermomechanical contact of bodies in the course of their frictional heating and the phenomenon of thermoelastic instability under friction was carried out in [88, 89, 92, 95]. It was established that wear processes are mostly affected not by volume temperature but by contact temperature. In a certain temperature range, the wear is minimal, but it increases sharply when a certain threshold value of temperature is reached. After that, plastic deformations and, as a rule, seizure and catastrophic fracture take place.

Wear as surface fracture is closely related to the roughness and waviness of surfaces, nominal, contour, and actual contact domains, and the tendency of materials toward crack initiation [2, 10, 102, 103].

In the course of technological treatment, microrelief is formed on the surface of materials. When two such surfaces interact, their contact is not continuous but discrete. The sum of these contact areas forms the actual contact domain. Its importance is explained by the fact that it determines the area of force interaction and the size of the domain of heat sources. The size of the actual contact domain depends on the microgeometry of the surfaces, i.e., on their roughness and waviness. To take roughness into account, different approaches are used. According to historically the first of these approaches [104], one introduces a combined elastic base and assumes that, along with general elastic deformation, there appear local deformations attributable to roughness. This model is called integral or continual. This approach was used in numerous works [17, 41, 54, 105], where, for the most part, a linear local dependence of additional displacement on contact stresses was used. Experimental and theoretical investigations of the deformation of rough surfaces showed that local strains are a power function of contact stresses [1, 7, 19, 102].

According to the discrete model, roughness is taken into account as a collection of individual protuberances. An analysis of profilograms enables one to establish certain regularities that characterize the shape and size of microprotuberances. To simplify solutions, one usually assumes that these protuberances have simple geometric shapes for which solutions of contact problems in the theory of elasticity and plasticity are known. Mathematical models were constructed and the corresponding calculations were made for rough surfaces in the case of spherical segments, cylinders with plane and convex bases, cones, bars, pyramids, ellipsoids with lugs

located at the same or different levels, etc. [7, 19, 102, 106]. It is most often assumed that the distribution of contact pressures under each individual protuberance corresponds to the Hertz distribution. With the use of this model, a relationship between the load and the settling of the rough layer was established and the actual contact domain and many other characteristics were computed.

In practically all works where the discrete model was used, the mutual influence of microprotuberances is neglected. This works only in the case of low loads of the contact couple and small contact densities. However, if these conditions are not satisfied (e.g., in the case of a saturated contact), this neglect leads to erroneous results. The first attempt to take this interaction into account [7] was based on the solution [11] of the problem of contact of two dies with a plane base (plane problem). A further advance in this direction was made in [107, 108], where the contact problems of interaction between an elastic half-space and two cylindrical dies of the same radius with plane base were solved and an important approximate formula relating the displacement to the applied load was obtained. The mutual influence of microprotuberances was evaluated by means of comparison of deflection under the dies with the solution [11] for a single cylindrical die with plane base.

In the same formulation, by the method of spliced asymptotic expansions, a solution in quadratures was obtained for two cylindrical dies with plane bases of different radii [109] and for dies with plane elliptic bases [110]. In these problems, dies with plane bases are considered, and, therefore, the actual and contour contact domains are independent of the load applied. In [111], the plane contact problem of interaction of two dies with a smooth profile in the form of a Legendre polynomial was studied. In this case, the actual contact domain depends considerably on the load, but the following disadvantage remains: as the distance between two lugs changes, the shape of microprotuberances also changes. A proper selection of the shape of two dies made it possible to construct analytic solutions of plane problems of interaction of two wedges and parabolas with a half-plane and analyze the dependence of this interaction on the load and physicomechanical and geometric characteristics [112, 113].

In the application of the solutions mentioned above to the analysis of the effect of surface roughness, an important role is played by the results of investigations of the stress-strain state of an elastic isotropic half-space loaded by a constant or Hertz distribution at circular domains located at the nodes of a hexagonal grid with certain period carried out in [19, 114].

The contact problems of interaction of surfaces of regular structure (periodic problems) can readily be analyzed and solved. Plane problems of this type were considered in [103, 104, 115–117]. In these works, periodic lugs of rectangular, sinusoidal, and wedge-like shape were investigated. Kuznetsov [116] also analyzed the stress-strain state of an elastic half-plane under the action of a periodic die. He showed that the lines of constant tangential stresses are stretched parallel to the contact surface as the distance decreases. In recent years, the theory of random functions has extensively been used for the description of surface roughness [118, 119].

Technological treatment of the parts of a friction assembly (as well as their interaction with the environment) changes the surface properties of contacting bodies. The deformation of rubbing bodies and high frictional temperature stimulate the diffusion of the intercontact medium into the surface layers and the diffusive exit of elements dissolved in the body to its surface. The appearance of surface inhomogeneities is explained by the localization of the maximum of heating temperature both on the contact surface and at a certain depth under it [120–125]. Furthermore, the quality, reliability, and durability of friction assemblies are guaranteed by technological methods and antifriction and wear-resistant coatings. In general, it is possible to endow contacting bodies with special surface properties both by the application of protective coatings and by proper transformations using mechanical, physical, and chemical processes, diffusion saturation, and heat treatment. Theoretical and practical aspects of these phenomena are studied in detail and described in [9, 19, 32, 120, 126–131].

Thus, the mathematical modeling of contact phenomena that takes into account surface inhomogeneity and a significant difference in the properties of surface layers is an important and urgent problem. One of the methods of its solution is connected with the model of a body with coating whose stress-strain state is described with the use of the theory of thin-walled structural elements [54, 67, 75, 132, 133]. This enables one to take into ac-

count anisotropy and inhomogeneity over the thickness as well as conditions of the contact of the coating and the base. This model is also convenient for problems of optimization of the thickness of coatings, which, in many cases, is decisive for an increase in the wear resistance and rigidity of contact assemblies. Numerous contact problems for infinite, semiinfinite, and finite patches and coatings were investigated in [54, 67]. Different versions of the theory of coatings were developed by the asymptotic approximation of a solution for an isotropic solid sphere, including the cases of cylindrical and spherical shells [67]. In [132, 133], the contact problem of interaction of a rigid die with an infinite Kirchhoff beam lying on an elastic half-plane was considered. In [132], the problem was reduced to dual integral equations; in [133], it was reduced to a Fredholm integral equation. Multilayer structures of coatings and the influence of the theories of thin-walled elements on the distribution of contact stresses under a die and between the coating and the base were analyzed in [4, 75, 82]. The wear of thin coatings on solids was considered in [75, 129]. The works [105, 132–134] are devoted to the problem of design of coatings for structural elements of triboengineering from the standpoint of stability of their equilibrium. Lamination, surface loads (an analog of contact stresses), temperature, and significant difference in the physico-mechanical characteristics of the main material and coatings were pointed out as characteristic features of coatings.

The present review shows that, despite substantial progress in the field of contact interaction of thin-walled solids, there are many problems in tribomechanics awaiting investigation and solution. In particular, one of these problems is connected with the construction of specific theoretical and adequate mathematical models for studying the wear of friction assemblies, especially in the case where a thin-walled element works in a friction couple. The mutual influence of individual microprotuberances during the contact of rough surfaces and its importance for the computation of the actual contact domain, the convergence of surfaces, and the stress distribution in the bodies have been inadequately studied, and the effect of residual stresses in bodies with coatings on the processes of contacting, friction, and wear has not been clarified thus far.

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