

Analysis of the Stress-Strain State of an Anisotropic Plate with an Elliptic Hole and Thin Elastic Inclusions

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Abstract—We solve the problem of determining the stress-strain state of an anisotropic plate with an elliptic hole and a system of thin rectilinear elastic inclusions. We assume that there is a perfect mechanical contact between the inclusions and the plate. We deal with a more precise junction model with the flexural rigidity of inclusions taken into account. (The tangential and normal stresses, as well as the derivatives of the displacements, experience a jump across the line of contact.) The solution of the problem is constructed in the form of complex potentials automatically satisfying the boundary conditions on the contour of the elliptic hole and at infinity. The problem is reduced to a system of singular integral equations, which is solved numerically. We study the influence of the rigidity and geometry parameters of the elastic inclusions on the stress distribution and value on the contour of the hole in the plate. We also compare the numerical results obtained here with the known data.

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1. INTRODUCTION

The problems of analysis of bodies with inclusions are of great importance in connection with wide application of composite materials and structures strengthened and reinforced with thin-walled elements. In several papers concerning elasticity problems for bodies with inclusions (see [1, 2] and the surveys therein), the inclusion was considered as a one-dimensional continuum (rod) working only in tension and compression. (Under this approach, the inclusion was modeled by a line across which the tangential stresses experience a jump but the normal stresses and the displacements vary continuously). In several papers, it was assumed that the inclusion is a plate with negligibly small flexural rigidity. In the third approach, the inclusion was considered as a plate and the model was constructed under the assumption that one of its linear dimensions is significantly smaller than the other.

In the present paper, we first study the stress-strain states of an infinite anisotropic plate with an elliptic hole and a system of rectilinear thin elastic inclusions. The elastic inclusion is considered in the framework of the third model as a plate one of whose linear dimensions (width) is significantly smaller than the other. The boundary conditions on the inclusion are written under the assumption that the tangential and normal stresses, as well the derivatives of the displacements, experience a jump across the contact line. We construct special integral representations of the solution of the problem determined from the boundary conditions along the contact lines and automatically satisfying the boundary conditions on the contour of the elliptic hole and at infinity. The problem is reduced to a system of singular integral equations, and an algorithm for solving this problem numerically is given. We present numerical results illustrating the influence of the geometry and rigidity parameters (the flexural rigidity and the tension-compression rigidity) of the elastic inclusions on the stress distribution on the contour of the hole in the plate. In the case of an elliptic hole degenerating into a rectilinear cut, we calculate the values of the stress intensity factors at the vertices of the cut. We compare the numerical results with the data obtained on the basis of the simplified model representing an elastic inclusion as a one-dimensional continuum working only in tension and compression.

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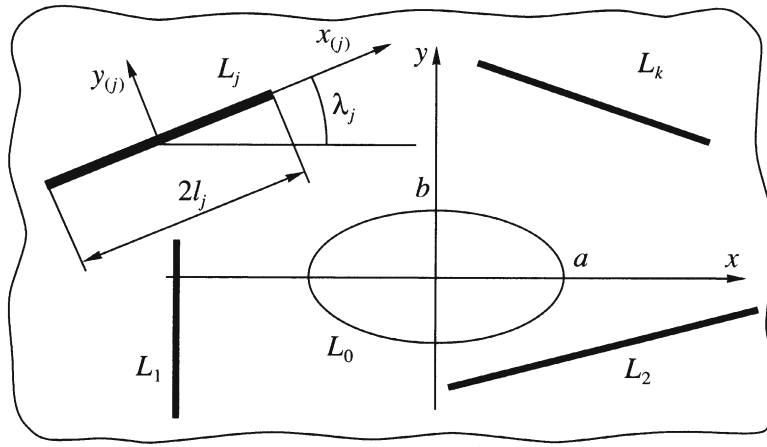


Fig. 1.

2. STATEMENT OF THE PROBLEM

Consider an infinite rectilinear anisotropic plate of thickness h with an elliptic hole $L_0 = \{(x/a)^2 + (y/b)^2 = 1\}$ and a system M of rectilinear thin elastic inclusions. In the plate, we introduce a general rectilinear coordinate system xOy . For the j th inclusion ($j = \overline{1, M}$), we introduce a local coordinate system $x_{(j)}O_{(j)}y_{(j)}$, where $O_{(j)}$ is the inclusion geometric center, the axis $O_{(j)}x_{(j)}$ is directed along the j th inclusion, and the angle between the axis $O_{(j)}x_{(j)}$ and the axis Ox is denoted by λ_j (Fig. 1). We assume that the j th inclusion has length $2l_j$, width $2b_j$, and thickness h . We denote the interval $[-l_j, l_j]$ of the axis $O_{(j)}x_{(j)}$ by L_j and assign the index (j) to the quantities characterizing the j th inclusion. By the indices $(+)$ and $(-)$ we denote the boundary values of functions as $y_{(j)} \rightarrow +0$ and $y_{(j)} \rightarrow -0$, respectively. The plate is loaded by external forces $X_n + iY_n$ applied on the hole contour and by forces σ_x^∞ , σ_y^∞ , and τ_{xy}^∞ at infinity. We assume that the plate is in a generalized plane stress state and a perfect mechanical contact with the plate material is realized on the inclusion edges.

The conditions of contact of the inclusions with the plate can be represented in the form

$$(\sigma_x^{(j)} - i\tau_{xy}^{(j)})|_{y_{(j)}=\pm b_j} = (\sigma_y - i\tau_{xy})|_{y_{(j)}=\pm b_j}, \quad (2.1)$$

$$(u'_{(j)} + iv'_{(j)})|_{y_{(j)}=\pm b_j} = (u' + iv')|_{y_{(j)}=\pm b_j} - i\omega_j, \quad (2.2)$$

where ω_j is the angle of rotation of the j th inclusion as a rigid unit.

3. DERIVATION OF THE SYSTEM OF EQUATIONS OF THE PROBLEM

We treat the inclusion as an anisotropic plate whose principal axes of anisotropy are directed along the axes $O_{(j)}x_{(j)}$ and $O_{(j)}y_{(j)}$, respectively. The stress-strain state of the j th inclusion can be described by the analytic functions $\Phi_{\nu j}(z_{\nu j})$, where $z_{\nu j} = x_{(j)} + \mu_{\nu j}y_{(j)}$ and the $\mu_{\nu j}$ are the roots with positive imaginary parts of the corresponding characteristic equation [3]:

$$(\sigma_x^{(j)}, \sigma_y^{(j)}, \tau_{xy}^{(j)}) = 2 \operatorname{Re} \left[\sum_{\nu=1}^2 (\mu_{\nu j}^2, 1, -\mu_{\nu j}) \Phi_{\nu j}(z_{\nu j}) \right], \quad (3.1)$$

$$(u'_{(j)}, v'_{(j)}) = 2 \operatorname{Re} \left[\sum_{\nu=1}^2 (p_{\nu j}, q_{\nu j}) \Phi_{\nu j}(z_{\nu j}) \right] + (0, \omega_j), \quad (3.2)$$

$$p_{\nu j} = a_{11}^{(j)} \mu_{\nu j}^2 - a_{16}^{(j)} \mu_{\nu j} + a_{12}^{(j)}, \quad q_{\nu j} = a_{12}^{(j)} \mu_{\nu j} - a_{22}^{(j)} \mu_{\nu j}^{-1} - a_{26}^{(j)},$$

where the $a_{ik}^{(j)}$ are the strain coefficients of the material of the j th inclusion.

Assuming that the inclusions are thin in width, we use the expansion of the complex potentials $\Phi_{\nu j}(z_{\nu j})$ in Taylor series in powers of b_j in a neighborhood of the point t on the real axis $O_{(j)}x_{(j)}$

and, neglecting the terms of higher order of magnitude compared with b_j , obtain

$$(\sigma_y^{(j)\pm}(t), \tau_{xy}^{(j)\pm}(t)) = 2 \operatorname{Re} \left[\sum_{\nu=1}^2 (1, -\mu_{\nu j}) \chi_{\nu j}^{\pm}(t) \right], \quad (3.3)$$

$$(u_{(j)}^{\pm}(t), v_{(j)}^{\pm}(t)) = 2 \operatorname{Re} \left[\sum_{\nu=1}^2 (p_{\nu j}, q_{\nu j}) \chi_{\nu j}^{\pm}(t) \right], \quad (3.4)$$

$$\chi_{\nu j}^{\pm}(t) = \Phi_{\nu j}(t) \pm b_j \mu_{\nu j} \Phi'_{\nu j}(t), \quad t \in L_j \quad (j = \overline{1, M}).$$

The stresses and the derivatives of the displacements in the plate have the form [3]

$$(\sigma_x, \sigma_y, \tau_{xy}) = 2 \operatorname{Re} \left[\sum_{\nu=1}^2 (\mu_{\nu}^2, 1, -\mu_{\nu}) \Phi_{\nu}(z_{\nu}) \right], \quad (3.5)$$

$$(u', v') = 2 \operatorname{Re} \left[\sum_{\nu=1}^2 (p_{\nu}, q_{\nu}) \Phi_{\nu}(z_{\nu}) \right] + (0, \omega_0), \quad (3.6)$$

$$p_{\nu} = a_{11}\mu_{\nu}^2 - a_{16}\mu_{\nu} + a_{12}, \quad q_{\nu} = a_{12}\mu_{\nu} - a_{22}\mu_{\nu}^{-1} - a_{26}, \quad z_{\nu} = x + \mu_{\nu}y,$$

where μ_{ν} and a_{ik} are parameters of the anisotropic material of the plate and ω_0 is the angle of rotation of the plate as a rigid unit.

By analogy with [4, 5], we seek the complex potentials $\Phi_{\nu}(z_{\nu})$ describing the stress-strain state of the anisotropic plate in the form

$$\Phi_{\nu}(z_{\nu}) = \sum_{i=0}^2 \Phi_{\nu i}(z_{\nu}). \quad (3.7)$$

Here the functions $\Phi_{\nu 0}(z_{\nu})$ determine the well-known basic stress state due to the external load on the hole contour and at infinity in the plate without elastic inclusions, and the functions $\Phi_{\nu 1}(z_{\nu})$ and $\Phi_{\nu 2}(z_{\nu})$ describe the perturbed stress state due to the elastic inclusions.

We start from the solution of the problem on the action of a lumped force at an internal point τ of an infinite anisotropic plate that has an elliptic hole L_0 free from external forces and is load-free at infinity [3]:

$$\Phi_{\nu}^*(z_{\nu}) = \frac{1}{\omega'_{\nu}(\zeta_{\nu})} \left[\frac{A_{\nu}}{\zeta_{\nu} - \eta_{\nu}} + \frac{l_{\nu} \bar{A}_1}{\zeta_{\nu}(\zeta_{\nu} \bar{\eta}_1 - 1)} + \frac{n_{\nu} \bar{A}_2}{\zeta_{\nu}(\zeta_{\nu} \bar{\eta}_2 - 1)} \right], \quad (3.8)$$

$$z_{\nu} = \omega_{\nu}(\zeta_{\nu}) = \frac{a - i\mu_{\nu}b}{2} \zeta_{\nu} + \frac{a + i\mu_{\nu}b}{2} \zeta_{\nu}^{-1}, \quad |\zeta_{\nu}| > 1,$$

$$\zeta_{\nu} = \zeta_{\nu}(z_{\nu}) = \frac{z_{\nu} + \sqrt{z_{\nu}^2 - (a^2 + \mu_{\nu}^2 b^2)}}{a - i\mu_{\nu}b}, \quad \zeta_{\nu}(\infty) = \infty,$$

$$\eta_{\nu} = \zeta_{\nu}(\tau_{\nu}) = \frac{\tau_{\nu} + \sqrt{\tau_{\nu}^2 - (a^2 + \mu_{\nu}^2 b^2)}}{a - i\mu_{\nu}b}, \quad \tau_{\nu} = \operatorname{Re} \tau + \mu_{\nu} \operatorname{Im} \tau,$$

$$l_{\nu} = \frac{\mu_{3-\nu} - \bar{\mu}_1}{\mu_{\nu} - \mu_{3-\nu}}, \quad n_{\nu} = \frac{\mu_{3-\nu} - \bar{\mu}_2}{\mu_{\nu} - \mu_{3-\nu}} \quad (\nu = 1, 2).$$

Using the superposition principle, we seek the functions $\Phi_{\nu k}(z_{\nu})$ ($k = 1, 2$) in the form

$$\begin{aligned} \Phi_{\nu k}(z_{\nu}) &= \frac{1}{\omega'_{\nu}(\zeta_{\nu})} \int_L \left[\frac{A_{\nu k}(\tau)}{\zeta_{\nu} - \eta_{\nu}} + \frac{l_{\nu} \bar{A}_{1k}(\tau)}{\zeta_{\nu}(\zeta_{\nu} \bar{\eta}_1 - 1)} + \frac{n_{\nu} \bar{A}_{2k}(\tau)}{\zeta_{\nu}(\zeta_{\nu} \bar{\eta}_2 - 1)} \right] ds \\ &= \frac{1}{2\pi i \omega'_{\nu}(\zeta_{\nu})} \int_L \left[\frac{\Omega_{\nu k}(\tau) d\tau_{\nu}}{\zeta_{\nu} - \eta_{\nu}} - \frac{l_{\nu} \bar{\Omega}_{1k}(\tau) d\bar{\tau}_1}{\zeta_{\nu}(1 - \zeta_{\nu} \bar{\eta}_1)} + \frac{n_{\nu} \bar{\Omega}_{2k}(\tau) d\bar{\tau}_2}{\zeta_{\nu}(1 - \zeta_{\nu} \bar{\eta}_2)} \right], \end{aligned} \quad (3.9)$$

$$\Omega_{\nu k}(\tau) = -\frac{2\pi i A_{\nu k}(\tau)}{M_{\nu}(\tau)}, \quad M_{\nu}(\tau) = \mu_{\nu} \sin \lambda(\tau) + \cos \lambda(\tau), \quad L = \bigcup_{j=1}^M L_j,$$

where $\Omega_{\nu k}(\tau) = \{\Omega_{\nu k}^{(j)}(\tau) | \tau \in L_j; j = \overline{1, M}; k, \nu = 1, 2\}$ are unknown complex functions on the contours L_j , $\lambda = \lambda(\tau) = \{\lambda_j(\tau) | \tau \in L_j, j = \overline{1, M}\}$, $\lambda_j(\tau)$ is the angle of inclination of the j th inclusion to the axis Ox , and $d\tau_\nu = M_\nu(\tau)ds$. (Here ds is the arc length differential on L .)

The functions $\Phi_{\nu k}(z_\nu)$ thus constructed automatically satisfy the conditions $X_n = Y_n = 0$ on the hole L_0 and decay at infinity. Hence the choice of the functions $\Phi_\nu(z_\nu)$ in the form (3.7), (3.9) ensures that the boundary conditions are satisfied on the hole contour L_0 and at infinity.

On the inclusion edges, the potentials $\Phi_\nu(z_\nu)$ must satisfy the boundary conditions [4, 5]

$$a_1(t)\Phi_1^\pm(t) + b_1(t)\overline{\Phi_1^\pm(t)} + \Phi_2^\pm(t) = F^\pm(t), \quad (3.10)$$

$$a_2(t)\Phi_1^\pm(t) + b_2(t)\overline{\Phi_1^\pm(t)} + \Phi_2^\pm(t) = W^\pm(t), \quad t \in L = \bigcup_{j=1}^M L_j, \quad (3.11)$$

$$\begin{aligned} a_1(t) &= a_0 \frac{M_1(t)}{M_2(t)}, \quad b_1(t) = b_0 \frac{\overline{M_1(t)}}{M_2(t)}, \quad a_0 = \frac{\mu_1 - \overline{\mu}_2}{\mu_2 - \overline{\mu}_2}, \\ b_0 &= \frac{\overline{\mu}_1 - \overline{\mu}_2}{\mu_2 - \overline{\mu}_2}, \quad a_2(t) = A_0 \frac{M_1(t)}{M_2(t)}, \quad b_2(t) = B_0 \frac{\overline{M_1(t)}}{M_2(t)}, \\ A_0 &= \frac{\overline{p}_2 q_1 - p_1 \overline{q}_2}{\overline{p}_2 q_2 - p_2 \overline{q}_2}, \quad B_0 = \frac{\overline{p}_2 \overline{q}_1 - \overline{p}_1 \overline{q}_2}{\overline{p}_2 q_2 - p_2 \overline{q}_2}, \\ F^\pm(t) &= \frac{\tau_{xy}^\pm(t) + \overline{\mu}_2 \sigma_y^\pm(t)}{(\mu_2 - \overline{\mu}_2)M_2(t)}, \quad W^\pm(t) = \frac{\overline{p}_2 v'^\pm(t) - \overline{q}_2 u'^\pm(t)}{(\overline{p}_2 q_2 - p_2 \overline{q}_2)M_2(t)}, \end{aligned}$$

where $\tau_{xy}^\pm(t)$, $\sigma_y^\pm(t)$, $v'^\pm(t)$, and $u'^\pm(t)$ are the unknown contact stresses and the derivatives of the plate displacements at the inclusion edges L_j .

The following conditions must be satisfied on the inclusion edges [5]:

$$\begin{aligned} a_1(t)\Omega_{11}(t) + b_1(t)\overline{\Omega_{11}(t)} + \Omega_{21}(t) &= 0, \\ a_2(t)\Omega_{12}(t) + b_2(t)\overline{\Omega_{12}(t)} + \Omega_{22}(t) &= 0, \quad t \in L = \bigcup_{j=1}^M L_j. \end{aligned} \quad (3.12)$$

In Eqs. (3.10) and (3.11), we subtract the functions with the superscript $(-)$ from the functions with the superscript $(+)$ and use relations (3.12) and the Sokhotskii–Plemelj formulas to obtain

$$\begin{aligned} a^*(t)\Omega_{12}(t) + b^*(t)\overline{\Omega_{12}(t)} &= F^+(t) - F^-(t), \\ a^*(t)\Omega_{11}(t) + b^*(t)\overline{\Omega_{11}(t)} &= -[W^+(t) - W^-(t)], \\ a^*(t) &= a_1(t) - a_2(t), \quad b^*(t) = b_1(t) - b_2(t), \quad t \in L. \end{aligned} \quad (3.13)$$

To simplify the obtained relations, we assume that the plate and the inclusions are made of an orthotropic material; i.e., $\mu_\nu = i\beta_\nu$ and $\mu_{\nu j} = i\beta_{\nu j}$ (Here β_ν and $\beta_{\nu j}$ are material parameters of the plate and the inclusions, respectively.)

From Eqs. (3.13), using relations (2.1), (2.2), (3.3), and (3.4), we obtain

$$\begin{aligned} \sum_{\nu=1}^2 c_{k\nu} A_\nu(t) + c_{k,\nu+2} B_\nu(t) &= R_k \varphi_1(t) + (-1)^k H_{3-k} \varphi_2(t) \quad (k = 1, 2), \\ \sum_{\nu=1}^2 c_{k+2,\nu} A_\nu(t) + c_{k+2,\nu+2} B_\nu(t) &= R_k \psi_1(t) + (-1)^k H_{3-k} \psi_2(t), \end{aligned} \quad (3.14)$$

$$\begin{aligned}
\frac{c_{k\nu}}{\beta_{\nu j}} &= s_{\nu j} \left[-\frac{T_{k1}}{p_2} + (-1)^{k+1} \frac{T_{3-k,2}}{s_2} \right], \quad T_{\nu 2} = 2b_j Q_\nu \cos \lambda_j, \\
\frac{c_{k,\nu+2}}{\beta_{\nu j}} &= p_{\nu j} \left[\frac{T_{k2}}{p_2} + (-1)^{k+1} \frac{T_{3-k,1}}{s_2} \right], \quad T_{\nu 1} = 2b_j Q_\nu \sin \lambda_j, \\
\frac{c_{k+2,\nu}}{\beta_{\nu j}^2} &= \left[(-1)^{k+1} \frac{T_{2,3-k}}{\beta_2} - \frac{T_{1k}}{\beta_2} \right], \quad Q_1 = \operatorname{Re} \left[\frac{1}{M_2(t)} \right], \\
\frac{c_{k+2,\nu+2}}{\beta_{\nu j}^2} &= \left[\frac{T_{2k}}{\beta_2} + (-1)^{k+1} \frac{T_{1,3-k}}{\beta_2} \right], \quad Q_1 = \operatorname{Im} \left[\frac{1}{M_2(t)} \right] \\
R_1 &= \operatorname{Re}[a^*(t) + b^*(t)], \quad H_1 = \operatorname{Re}[a^*(t) - b^*(t)], \quad s_\nu = -iq_\nu, \\
R_2 &= \operatorname{Im}[a^*(t) + b^*(t)], \quad H_2 = \operatorname{Im}[a^*(t) - b^*(t)], \quad s_{\nu j} = -iq_{\nu j}, \\
\varphi_1(t) &= \operatorname{Re} \Omega_{11}(t), \quad \psi_1(t) = \operatorname{Re} \Omega_{12}(t), \quad A_\nu(t) = \operatorname{Re} \Phi'_{\nu j}(t), \\
\varphi_2(t) &= \operatorname{Im} \Omega_{11}(t), \quad \psi_2(t) = \operatorname{Im} \Omega_{12}(t), \quad B_\nu(t) = \operatorname{Im} \Phi'_{\nu j}(t).
\end{aligned}$$

By solving system (3.14), we express the functions $A_\nu(t)$ and $B_\nu(t)$ via the functions $\varphi_\nu(t)$ and $\psi_\nu(t)$:

$$\begin{aligned}
A_k(t) &= \sum_{\nu=1}^2 c_{k\nu}^* \varphi_\nu(t) + c_{k,\nu+2}^* \psi_\nu(t), \\
B_k(t) &= \sum_{\nu=1}^2 c_{k+2,\nu}^* \varphi_\nu(t) + c_{k+2,\nu+2}^* \psi_\nu(t) \quad (k = 1, 2).
\end{aligned} \tag{3.15}$$

Since the coefficients c_{ij}^* are very cumbersome, we do not present them here.

In the boundary conditions (3.10) and (3.11), we add the functions with the superscript $(-)$ to the functions with the superscript $(+)$ and use (3.12) and the Sokhotskii–Plemelj formulas to obtain the fundamental system of the singular integral equations of the problem:

$$\sum_{k=1}^2 \int_L [K_{1,2k-1}^*(t, \tau) \Omega_{1k}(\tau) + K_{1,2k}^*(t, \tau) \overline{\Omega_{1k}(\tau)}] ds = f_1(t), \quad t \in L, \tag{3.16}$$

$$\sum_{k=1}^2 \int_L [K_{2,2k-1}^*(t, \tau) \Omega_{1k}(\tau) + K_{2,2k}^*(t, \tau) \overline{\Omega_{1k}(\tau)}] ds + \Lambda_0 = f_2(t),$$

$$K_{\nu m}^*(t, \tau) = K_{\nu m}(t, \tau) - \Delta(t, \tau) k_{\nu m}(t, \tau) \quad (m = \overline{1, 4}; \nu = 1, 2),$$

$$\begin{aligned}
K_{\nu, 2k-1}(t, \tau) ds &= \frac{a_\nu(t)}{\pi i \omega'_1(\zeta_1)} \left[\frac{d\tau_1}{\eta_1 - \zeta_1} + \frac{n_1 \overline{b_k(\tau)} d\tau_2}{\zeta_1(1 - \zeta_1 \overline{\eta_2})} \right] \\
&+ \frac{b_\nu(t)}{\pi i \overline{\omega'_1(\zeta_1)}} \left[\frac{\overline{l_1} d\tau_1}{\zeta_1(1 - \overline{\zeta_1} \eta_1)} - \frac{\overline{n_1} a_k(\tau) d\tau_2}{\zeta_1(1 - \overline{\zeta_1} \eta_2)} \right] + \frac{1}{\pi i \omega'_2(\zeta_2)} \left[-\frac{a_k(\tau) d\tau_2}{\eta_2 - \zeta_2} + \frac{n_2 \overline{b_k(\tau)} d\tau_2}{\zeta_2(1 - \zeta_2 \overline{\eta_2})} \right],
\end{aligned}$$

$$\begin{aligned}
K_{\nu, 2k}(t, \tau) ds &= \frac{a_\nu(t)}{\pi i \omega'_1(\zeta_1)} \left[\frac{n_1 \overline{a_k(\tau)} d\tau_2}{\zeta_1(1 - \zeta_1 \overline{\eta_2})} + \frac{l_1 d\tau_1}{\zeta_1(1 - \zeta_1 \overline{\eta_1})} \right] \\
&+ \frac{b_\nu(t)}{\pi i \overline{\omega'_1(\zeta_1)}} \left[-\frac{d\tau_1}{\overline{\eta_1} - \overline{\zeta_1}} - \frac{\overline{n_1} b_k(\tau) d\tau_2}{\zeta_1(1 - \overline{\zeta_1} \eta_2)} \right] + \frac{1}{\pi i \omega'_2(\zeta_2)} \left[-\frac{b_k(\tau) d\tau_2}{\eta_2 - \zeta_2} - \frac{l_2 d\tau_1}{\zeta_2(1 - \zeta_2 \overline{\eta_1})} + \frac{n_2 \overline{a_k(\tau)} d\tau_2}{\zeta_2(1 - \zeta_2 \overline{\eta_2})} \right],
\end{aligned}$$

$$\Delta(t, \tau) = 0, \quad \text{if } t < \tau; \quad \Delta(t, \tau) = 1, \quad \text{if } t > \tau,$$

$$k_{1\nu}(t, \tau) = h_1(t) + i^{2\nu+1} h_2(t), \quad k_{1,2+\nu}(t, \tau) = h_3(t) + i^{2\nu+1} h_4(t),$$

$$k_{2\nu}(t, \tau) = h_1^*(t) + i^{2\nu+1} h_2^*(t), \quad k_{2,2+\nu}(t, \tau) = h_3^*(t) + i^{2\nu+1} h_4^*(t),$$

$$h_m(t) = -\frac{1}{M_2(t)} \left(\frac{i}{\beta_2} D_m + G_m \right), \quad h_m^*(t) = \frac{1}{M_2(t)} \left(\frac{i}{s_2} D_m^* + \frac{G_m^*}{p_2} \right),$$

$$D_m = \sum_{\nu=1}^2 \beta_{\nu j} (\operatorname{Im} c_{\nu m}^* + \operatorname{Re} c_{\nu+2, m}^*), \quad G_m = \sum_{\nu=1}^2 (\operatorname{Re} c_{\nu m}^* - \operatorname{Im} c_{\nu+2, m}^*),$$

$$D_m^* = \sum_{\nu=1}^2 s_{\nu j} (\operatorname{Im} c_{\nu m}^* + \operatorname{Re} c_{\nu+2, m}^*), \quad G_m^* = \sum_{\nu=1}^2 p_{\nu j} (\operatorname{Re} c_{\nu m}^* - \operatorname{Im} c_{\nu+2, m}^*),$$

$$f_{\nu}(t) = -2[a_{\nu}(t)\Phi_{10} + b_{\nu}(t)\overline{\Phi}_{10} + \Phi_{20}], \quad \Lambda_0 = \frac{i\omega_{0j}}{s_2 M_2(t)}.$$

In addition to the system of singular integral equations (3.16), the equilibrium condition and the displacement uniqueness conditions must be satisfied when going around each of the inclusions:

$$\int_L \Omega_{11}(\tau) ds = 0, \quad \int_L \Omega_{12}(\tau) ds = 0,$$

$$\operatorname{Re} \left\{ \int_L [\tau_1 - \tau_2 a_2(\tau) - \overline{\tau_2} \overline{b_2(\tau)}] \Omega_{12}(\tau) ds \right\} = 0, \quad \tau \in L = \bigcup_{j=1}^M L_j. \quad (3.17)$$

The system of singular integral equations (3.16), together with the additional conditions (3.17), gives a solution of the problem of determining the stress-strain state of an anisotropic plate with an elliptic hole and a system of M thin elastic rectilinear inclusions.

4. NUMERICAL REALIZATION AND COMPUTATIONAL EXAMPLES

We seek the unknown functions $\Omega_{1k}(\tau)$ ($k = 1, 2$) in the form

$$\Omega_{1k}(\tau) = \Omega_{1k}^0(\xi)(1 - \xi^2)^{-1/2}, \quad \tau \in L = \{\tau = \tau^j(\xi) | -1 < \xi < 1 \ (j = \overline{1, M})\},$$

where the $\Omega_{1k}^0(\xi)$ are bounded functions Hölder continuous on the interval $[-1, 1]$. Using the Gauss–Chebyshev quadrature formulas [6], we reduce the system of singular integral equations (3.16) with the additional conditions (3.17) to a system of linear algebraic equations for the approximate values of the unknown functions $\Omega_{1k}^0(\xi)$ at the Chebyshev nodes $\xi_i^{(j)} = \cos[(2i - 1)\pi]/(2N_j)$, $i = \overline{1, N_j}$; $j = \overline{1, M}$. Theoretical estimates of the convergence of this numerical method are given in [7].

Once the system of linear algebraic equations has been solved and $\Omega_{1k}^0(\xi)$ have been determined, we can calculate the values of the potentials $\Phi_{\nu}(z_{\nu})$ and the stresses in the plate by formula (3.5). If L_0 is a rectilinear cut of length $2a$ (for $b = 0$), we calculate the separation and shear stress intensity factors at the cut vertices:

$$K_1(c) = \lim_{t \rightarrow c} \sigma_n \sqrt{2\pi r}, \quad K_2(c) = \lim_{t \rightarrow c} \tau_n \sqrt{2\pi r}.$$

Here t is a point on the continuation of the cut beyond the vertex c along the tangent, and $r = |t - c|$.

In what follows, we present the results of computations for the plate with a circular hole $L_0 = \{(x/a)^2 + (y/a)^2 = 1\}$ made of an almost isotropic material ($E_1/E_2 = 1$, $\mu_1 = 1.004i$, $\mu_2 = 0.996i$); here E_1 and E_2 are the Young moduli of the plate material in the direction of the axes Ox and Oy , respectively. In the plate, there are two rectilinear elastic inclusions $L_j = \{\tau = \tau^j(\xi) = i(a + \Delta_j) + l_j \xi | -1 < \xi < 1; j = 1, 2\}$. At infinity, the plate is subjected to uniaxial tension under the forces p at an angle γ with the axis Ox (Fig. 2). For each j , to describe the j th inclusion, we introduce two parameters U_j^* and I_j^* where $U_j^* = E_1 a h / (E_{0j} F_j)$ is the relative longitudinal rigidity and $I_j^* = I_j / (h a^3)$ is the relative flexural rigidity. Here E_{0j} , $F_j = 2b_j h$, and $I_j = 2/3 h b_j^3$ are the Young modulus, the area, and the moment of inertia, respectively, of the transverse cross-section of the j th inclusion.

In Fig. 2, we show the influence of the relative flexural rigidity of the inclusions I_j^* on the tangential stress distribution σ_{ϑ}/p on the contour of the circular hole under the uniaxial tension of the plate along the axis Oy by the forces $\sigma_y^{\infty} = p$ ($\gamma = \pi/2$) for $\Delta_j/a = 0.2$, $l_j/a = 5.0$, and $U_j^* = 0.05$ with the parameters $I_j^* = 0; 10^{-4}; 10^{-3}; 10^{-2}; 10^{-1}$ (curves 1–5, respectively). The dotted line 1 corresponding to the value $I_j^* = 0$ (a momentless elastic inclusion) agrees with the results obtained in [8], where an

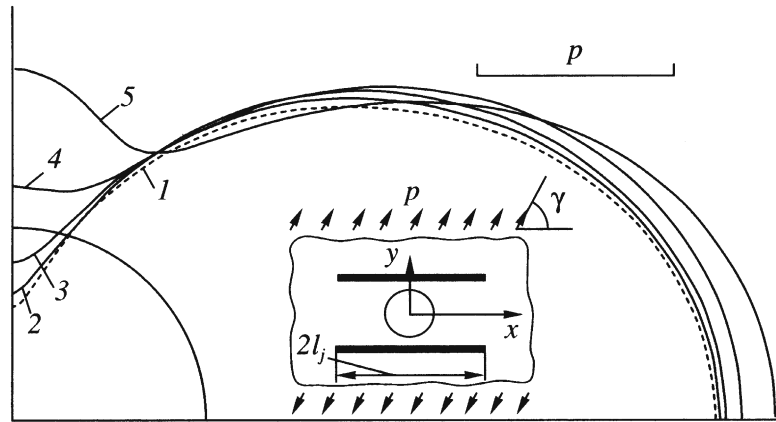


Fig. 2.

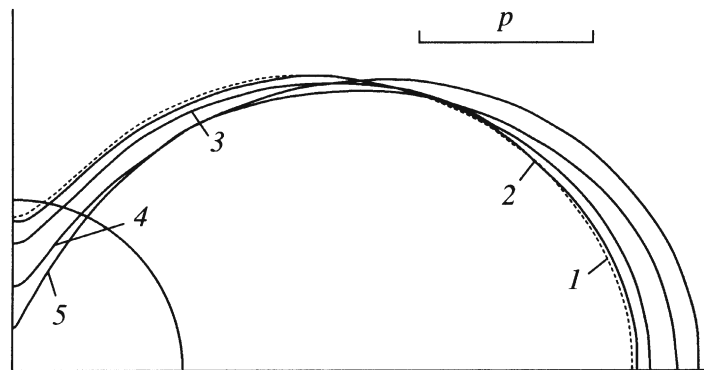


Fig. 3.

elastic inclusion is modeled by a “thin filament” working only in tension and compression. One can see that the flexural rigidity of inclusions significantly affects the tangential stress distribution on the hole contour (especially at the contour points nearest to the inclusions).

Figure 3 illustrates the influence of the relative tension-compression rigidity U_j^* of the inclusions on the tangential stress distribution σ_θ/p on the contour of the circular hole shown in Fig. 2, where $\Delta_j/a = 0.2$, $l_j/a = 5.0$, $I_j^* = 10^{-3}$, $\sigma_y^\infty = p$, $\gamma = \pi/2$, and $U_j^* = 0; 0.1; 0.5; 2.5; 50.0$ (curves 1–5, respectively). The dashed line (curve 1) corresponds to the case of inclusions that are absolutely rigid in tension-compression ($U_j^* = 0$). For $U_j^* = 50.0$, the stress diagram is close to the curve obtained for the case in which the plate does not contain any elastic inclusions.

Figure 4 illustrates the influence of the relative length l_j/a of elastic inclusions on the value of the stress concentration factor at the points $x=0$ and $y=\pm a$ of the circular hole contour in an isotropic plate (Fig. 2). The computations were performed for $\Delta_j/a = 0.2$, $U_j^* = 0.05$, and $I_j^* = 0; 10^{-4}; 5 \cdot 10^{-3}; 10^{-2}$ (curves 1–5, respectively); $\sigma_y^\infty = p$ and $\gamma = \pi/2$ (the lower curves); $\sigma_x^\infty = p$ and $\gamma = 0$ (the upper curves). For $\gamma = 0$, Fig. 4 presents only two curves for $I_j^* = 0; 10^{-2}$, because the other curves only slightly differ from those constructed above. The flexural rigidity of the inclusions affects the stress concentration in the plate more significantly if the loading is perpendicular to the inclusion lines. For $l_j/a > 1.5$, the stress concentration in the plate remains practically unchanged.

Consider the problem of determining the stress-strain state of an infinite plate with a cut of length $2a$ located along the axis Ox . (The contour L_0 becomes a cut if $b = 0$.) We perform the computations for an almost isotropic plate material ($E_1/E_2 = 1$, $\mu_1 = 1.004i$, $\mu_2 = 0.996i$) and two elastic inclusions $L_j = \{\tau = \tau^j(\xi) = l_j\xi + i^{2j-1}d| - 1 < \xi < 1; j = 1, 2\}$, where $l_j/a = 5.0$ and $U_j^* = 0.05$. At infinity, the plate is subjected to uniaxial tension $\sigma_y^\infty = p$ (Fig. 5). Figure 5 shows how the reduced stress intensity

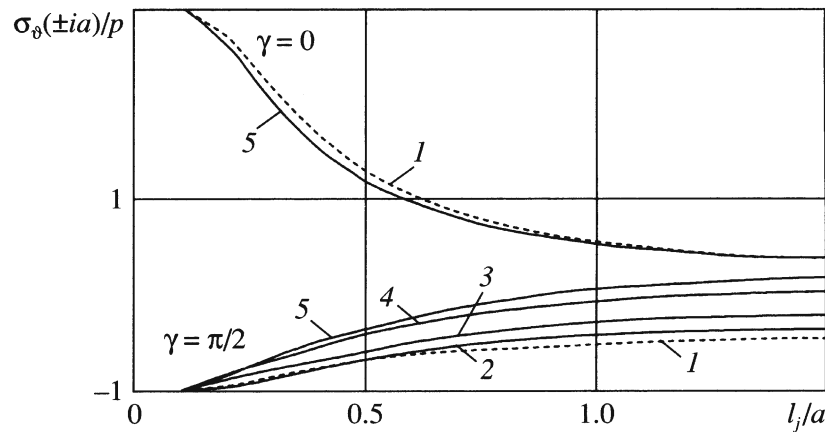


Fig. 4.

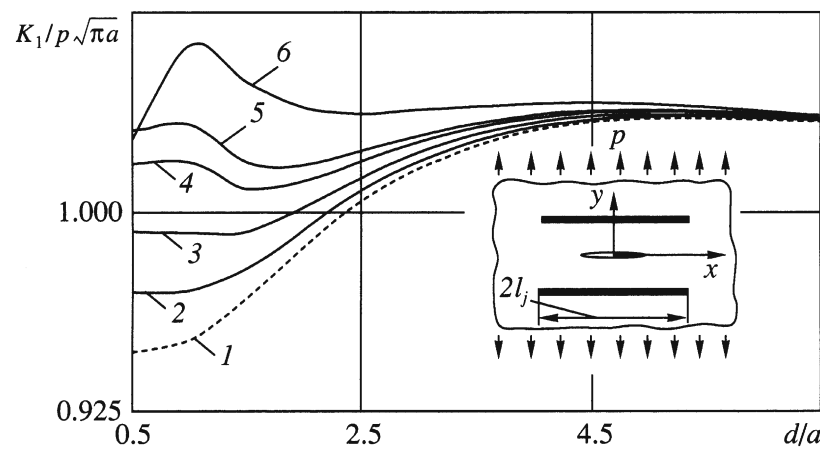


Fig. 5.

factor at the cut tips depends on the parameter d/a (where $2d$ is the distance between the inclusions) for various values $I_j^* = 0; 10^{-4}; 10^{-3}; 5 \cdot 10^{-3}; 10^{-2}; 10^{-1}$ (curves 1–6, respectively). The dashed line 1 corresponds to momentless elastic inclusions working only in tension-compression. As follows from Fig. 5, the flexural rigidity of the inclusions significantly affects the stress intensity factors at the cut tips for $d/a < 3$.

We compare the results obtained for a plate made of an almost isotropic material (Fig. 5) in the case where two elastic inclusions become cuts of length $2a$ ($U_j^* \rightarrow \infty, l_j/a = 1.0$). In Table 1, we present the results of computations of the stress intensity coefficients $K_1/p\sqrt{\pi a}$ at the vertices of the cuts L_j for various values of the parameter $2a/d$ and $N_j = 10$ ($j = 1, 2$) (where N_j is the number of interpolation nodes in the partition of the contours L_j). Note that as the number of interpolation points is doubled $N_j = 20$, the stress intensity factor $K_1/p\sqrt{\pi a}$ does not vary at the sixth decimal place. In Table 1, we compare the obtained results with the data given in [9]. One can see that only for $2a/d = 0.8$ the relative discrepancy from the data given in [9] is 0.152% (for the other values of the parameter $2a/d$, the relative error satisfies $|\delta| < 0.06\%$). In this case, the number of interpolation nodes on the contours L_j is equal to 10.

It follows from the preceding results that the rigidity characteristics of elastic inclusions (longitudinal and flexural rigidity) significantly affect the distribution and the values of stresses in a plate with a hole, as well as the stress intensity factors at the cut tips. The influence of the flexural rigidity of inclusions on the stress state in the plate is greater if the loading is perpendicular to the inclusion lines. The comparison of the computational results with the well-known results testifies that the method proposed here for

Table 1

$\frac{2a}{d}$	$K_1/p\sqrt{\pi a}$		$ \delta , \%$
	Data [9]	$N_j = 10$	
0.1	0.99500	0.995381	0.038
0.2	0.98198	0.982308	0.033
0.3	0.96299	0.962835	0.016
0.4	0.94010	0.939589	0.054
0.5	0.91535	0.915055	0.032
0.6	0.89080	0.891130	0.037
0.7	0.86851	0.868997	0.056
0.8	0.85052	0.849225	0.152

calculating the stress-strain state of an anisotropic plate with an elliptic hole (cut) and a system of thin rectilinear elastic inclusions (or cracks) is highly efficient and precise.

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