

(3.6) through (3.8). When $k_1 = k_2$, so that the vertical components of the velocity are equal at the two ends of the cylinder, (3.8) reduces to (3.6) or (3.7). The velocity fields (3.6), (3.7), (3.8) are realized when the geometry of the cylinder is such that $|BC|/|BB'| \geq 2$.

The flow velocity fields given above are to be understood as initial fields which will persist only during a small time interval.

Finally we point out that (2.4) through (2.10) can also be used in the solution of other problems; in particular (2.10) $|t| < 1$ can be used in the case of a half-space (Fig. 1) where the vertical component of the velocity is given along a portion of the parabola $|t| < 1$ (the imprint of a parabolic punch). Relation (2.6) can be used to construct the velocity field in a circular cylinder differing from that considered above.

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TWO CONTACT PROBLEMS OF THE THEORY OF ELASTICITY FOR A LAYER WITH A WINKLER-TYPE COATING

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UDC 539.3

A solution is presented for two contact problems of the theory of elasticity for a layer reinforced over its top boundary with a Winkler-type coating. A study is made of the two-dimensional problem, being the dynamic analog of the problem investigated in [4]. Here, we assume that inertial effects in the elastic band can be ignored compared to those in the coating. We will study an axisymmetric static problem. The principle of coincidence of the kernels of the integral equations of the two-dimensional and axisymmetric problems will be used to obtain an integral equation in the contact pressures. The unambiguous solvability of the latter is established in certain function spaces with any values of the characteristic geometric and physical parameters of the problem. An efficient approximate solution is also constructed.

1. We are examining an elastic band of considerable thickness H lying on a rigid base. The top boundary of the band is covered by a thin layer of continuously distributed vertical elastic rods ($0 \leq y \leq h$). We will assume that a rigid die of width $2a$ and having a flat base (Fig. 1) is pressed without friction into the surface of the top layer by a force $P(t) = P_0 + P_1 \exp(i\omega t)$. Due to the linearity of the problem being examined, its solution can be constructed as the superposition of the solution with $P(t) = P_0 = \text{const}$ obtained in [4] and the solution with $P(t) = P_1 \exp(i\omega t)$, which will be constructed in the present article.

The rod layer will be described by the following equations in the case of plane strain

$$\begin{aligned} u &= 0, \quad v = v(y, t), \quad \varepsilon_x = 0, \quad \gamma_{xy} = 0, \quad \sigma_y = R\varepsilon_y, \\ R &= 2G_1(1 - \nu_1)(1 - 2\nu_1)^{-1}, \quad \sigma_x = \nu_1(1 - \nu_1)^{-1}\sigma_y, \quad \tau_{xy} = 0, \\ \frac{1}{\mu^2} \frac{\partial^2 v}{\partial t^2} &= \frac{\partial^2 v}{\partial y^2}, \quad \mu^2 = \frac{R}{\rho_1}, \end{aligned} \quad (1.1)$$

where G_1 and ν_1 are the mechanical characteristics of the material of the coating and ρ_1 is

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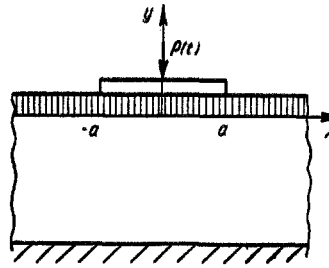


Fig.

its density; u and v are displacements of points of the rods along the y and x axes, respectively. From (1.1) we find that

$$v = \frac{1}{\mu^2} \int_0^y \int_0^{\xi} \frac{\partial^2 v}{\partial t^2} d\eta d\xi + A_1 y + A_2. \quad (1.2)$$

Here, $A_i = A_i(t)$ ($i = 1, 2$) are certain arbitrary functions.

Taking into consideration the boundary conditions of the following auxiliary problem

$$Rv'_y = -q \quad (y=0); \quad Rv'_y = -p \quad (y=h),$$

where $p(x, t)$ is the load acting on the upper boundary of the rod layer; $q(x, t)$ is the reaction load acting on the lower boundary from the direction of the elastic band; and taking into consideration as well the condition of contact of the layers

$$v = -v^* \quad (y=0),$$

where v^* is the displacement of the upper boundary of the elastic band along the y axis, we obtain

$$v(x, y, t) = \frac{1}{\mu^2} \int_0^y \int_0^{\xi} \frac{\partial^2 v}{\partial t^2} d\eta d\xi - \frac{1}{\rho_1 \mu^2} yq - v^*, \quad (1.3)$$

$$p = q - \rho_1 \int_0^h \frac{\partial^2 v}{\partial t^2} dy.$$

We will satisfy the condition of contact between the die and the surface of the rod layer by means of (1.3)

$$v = -\delta(t) \quad (y=h, |x| \leq a),$$

where $\delta(t)$ is the vertical displacement of the die under the influence of the force applied to it $P(t)$. Here, we obtain

$$\delta(t) = \frac{h}{\rho_1 \mu^2} q + v^* - \frac{1}{\mu^2} \int_0^h \frac{\partial^2 v}{\partial t^2} (h-y) dy \quad (|x| \leq a). \quad (1.4)$$

Retaining the inertial term in the Eq. (1.1) for the coating, we will ignore dynamic effects in the elastic band. This can be done if $P_1 \ll P_0$. In this case, the dynamic effects will be localized in the coating if we consider that the actual structure is viscoelastic (we ignore this low viscosity in the present examination). We will have the following expression for v^* [1]:

$$v^*(x, t) = \frac{1}{\pi\theta} \int_{-a}^a q(\xi, t) \left[-\ln \frac{|\xi - x|}{H} + \alpha_0 \right] d\xi \quad (|x| \leq a); \quad (1.5)$$

$$\theta = 4G_2(1 - \nu_2)(3 - 4\nu_2)^{-1}.$$

Here, G_2 and ν_2 are the elastic constants of the band; α_0 is a constant whose value is related to the boundary conditions on the lower boundary of the band. For example, in the case where the band lies without friction on a rigid base, $\alpha_0 = -0.352$. In the case of rigid fastening $\alpha_0 = -0.527$ with $\nu_2 = 0.3$.

Inserting (1.5) into (1.6), we calculate the integral equation of the problem for a two-layer base

$$\delta(t) = \frac{h}{\rho_1 \mu^2} q(x, t) + \frac{1}{\pi \theta} \int_{-a}^a q(\xi, t) \left[-\ln \frac{|\xi - x|}{H} + a_0 \right] d\xi - \frac{1}{\mu^2} \int_0^h (h - y) v''(x, y, t) dy \quad (|x| \leq a). \quad (1.6)$$

Also, the equation of motion of the die, of mass M , must be satisfied

$$\int_{-a}^a p(x, t) dx = P(t) - M \delta''(t). \quad (1.7)$$

We will assume that the displacements of points of the rod layer change little through the thickness. Then we can write

$$v(x, y, t) \approx -(1 - y h^{-1}) v^*(x, t) - \delta(t) y h^{-1},$$

from which, by integration, we obtain

$$\begin{aligned} \int_{-a}^a v''(x, y, t) dy &\approx -\frac{h}{2} \frac{\partial^2}{\partial t^2} [v^*(x, t) + \delta(t)]; \\ \int_{-a}^a (h - y) v''(x, y, t) dy &\approx -\frac{h^2}{6} \frac{\partial^2}{\partial t^2} [v^*(x, t) + \delta(t)]. \end{aligned} \quad (1.8)$$

Allowing for (1.8), we change (1.6) and (1.7) to the form

$$\frac{h}{\rho_1 \mu^2} q(x, t) + \frac{1}{\pi \theta} \int_{-a}^a \left[q(\xi, t) + \frac{h^2}{3 \mu^2} q''(\xi, t) \right] \left[-\ln \frac{|\xi - x|}{H} + a_0 \right] d\xi = \delta(t) - h^2 (6 \mu^2)^{-1} \delta''(t) \quad (|x| \leq a); \quad (1.9)$$

$$P(t) = \int_{-a}^a q(x, t) dx + M \delta''(t) + \rho_1 a h \delta''(t) + \rho_1 \frac{h}{2 \pi \theta} \int_{-a}^a \int_{-a}^a q''(\xi, t) \left[-\ln \frac{|\xi - x|}{H} + a_0 \right] d\xi dx. \quad (1.10)$$

Limiting ourselves to examination of the regime of harmonic vibration of the die, we represent the functions $q(x, t)$ and $\delta(t)$, entering into Eqs. (1.9) and (1.10), in the form

$$q(x, t) = q^*(x) \exp(i \omega t); \quad \delta(t) = \delta^* \exp(i \omega t). \quad (1.11)$$

Inserting (1.11) into (1.9) and (1.10) and separating the variables in these equations, we obtain an integral equation and the condition of motion of the die in amplitudes of the contact stress and the indenting force on the die. In dimensionless variables and with allowance for the notation

$$\begin{aligned} x &= a x', \quad \xi = a \xi', \quad q^*(x) \theta^{-1} = \varphi(x'), \\ c &= c^* (1 - \Omega^2), \quad \beta = \gamma c^* \left(1 + \frac{\Omega^2}{2} \right), \quad b = 3 \pi^{-1} \Omega^2 c^*, \\ c^* &= \rho_1 a \mu^2 (h \theta)^{-1}, \quad \Omega^2 = h^2 \omega^2 (3 \mu^2)^{-1}, \quad Q = P_1 (a \theta)^{-1}; \\ \alpha &= N - b [\pi \gamma + (a_0 + \ln \lambda + 1) N] - M \omega^2 \gamma \theta^{-1}; \\ \gamma &= \delta^* a^{-1}, \quad N = \int_{-1}^1 \varphi(x') dx', \quad \lambda = H a^{-1} \end{aligned}$$

we will have (primes omitted)

$$\varphi(x) + \frac{c}{\pi} \int_{-1}^1 \varphi(\xi) \left[-\ln \frac{|\xi - x|}{\lambda} + a_0 \right] d\xi = \beta \quad (|x| \leq 1); \quad (1.12)$$

$$Q = \alpha + \frac{b}{2} \int_{-1}^1 \varphi(x) [(1+x) \ln(1+x) + (1-x) \ln(1-x)] dx. \quad (1.13)$$

Equations (1.12) and (1.13) can be solved by a method such as that in [4]. However, in contrast to the integral equation in [4], the present equation, when $\omega > \sqrt{3} \mu h^{-1}$, may be a

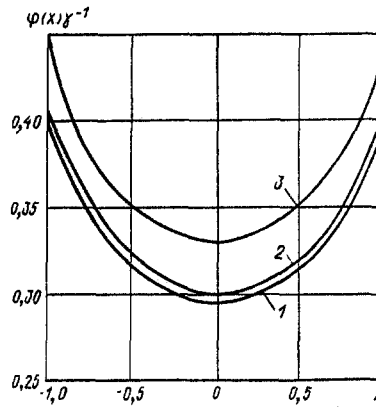


Fig. 2

denumerable set of positive characteristic numbers c_n which corresponds to a system of eigenfunctions $\{\varphi_n(x)\}$. This system is orthogonal and complete in the space of quadratically summable functions [5].

We will seek a solution to integral equation (1.12) in the form

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x). \quad (1.14)$$

Expanding the function $\pi^{-1}[-\ln|\xi - x| + \ln \lambda + \alpha_0]$ into a biorthogonal series

$$-\ln \frac{|\xi - x|}{\lambda} + \alpha_0 = \pi \sum_{n=1}^{\infty} \frac{1}{c_n} \varphi_n(x) \varphi_n(\xi) \quad (1.15)$$

with allowance for the representation

$$\beta = \sum_{n=1}^{\infty} b_n \varphi_n(x); \quad b_n = \beta \int_{-1}^1 \varphi_n(x) dx, \quad (1.16)$$

as well as the insertion of (1.14)-(1.16) into (1.12), we obtain

$$a_n = b_n c_n (c + c_n)^{-1}. \quad (1.17)$$

Series (1.14)-(1.16) reduce to their corresponding functions in the mean square space. It follows from (1.13), (1.14), and (1.17) that when $-c \rightarrow c_n$ ($n = 1, 2$) and with a prescribed force behind the die, the indentation under it will increase without limit, i.e. resonance will occur in this case.

It should be noted that the method in [5] can be used to find the eigenfunctions of integral operator (1.12) with a logarithmic kernel. We will not discuss this further here.

Let us illustrate the proposed algorithm numerically. We set $\alpha_0 = -0.352$; $M = 0$; $c^* = 1$; $\lambda = 16$. Figure 2 shows the relationship between the values of $\varphi(x)\gamma^{-1}$ and x calculated with $\Omega^2 = 0, 0.1$, and 0.3 (curves 1, 2, and 3, respectively). For these values of Ω^2 we have $Q\gamma^{-1} = 0.657, 0.624$, and 0.358 . To obtain 2% accuracy in the result, only the first four equations of system (2.8) in [5] should be used.

2. Let us examine an axisymmetric analog of the problem studied in [4]. To write its integral equation, we will use the following correspondence principle [1]. If we know the kernel of the integral equation of a two-dimensional contact problem of elasticity theory, represented in one of the forms

$$k(z) = \frac{1}{\pi} \int_0^{\infty} L(u) u^{-1} \cos uz du \quad \left(z = \frac{\xi - x}{H} \right); \quad s(z) = \frac{1}{\pi} \int_0^{\infty} L(u) \sin uz du,$$

then the kernel of the integral equation of the corresponding axisymmetric problem has the form

$$K(\xi, z) = \frac{1}{H} \int_0^{\infty} L(u) J_0(uz) J_0(u\xi) du; \quad \xi = \rho H^{-1}; \quad z = r H^{-1}, \quad (2.1)$$

while the function $L(u)$ is given by the expressions [1]

$$L(u) = \frac{\operatorname{ch} 2u - 1}{\operatorname{sh} 2u + 2u}; \quad L(u) = \frac{2\kappa \operatorname{sh} 2u - 4u}{2\kappa \operatorname{ch} 2u + 1 + \kappa^2 + 4u^2} \quad (\kappa = 3 - 4\nu_2)$$

respectively for the cases when the bottom layer (the band) lies without friction on a rigid base and when it is fastened to the base.

Thus, introducing the dimensionless variables and notation in accordance with the formulas

$$\rho = a\rho', \quad r = ar', \quad \lambda = Ha^{-1}, \quad \varphi(r') = q(r)\theta^{-1}, \quad Q = P(a^2\theta)^{-1};$$

$$f(r') = \delta(r)(k\theta)^{-1}, \quad c = a^2(k\theta H)^{-1}, \quad k_1 = (1 - 2\nu_1)h(2G_1(1 - \nu_1))^{-1},$$

where $\delta(r)$ is a function describing the form of the base of the die, we will have (primes omitted henceforth)

$$\varphi(r) + c \int_0^1 \rho \varphi(\rho) K\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right) d\rho = f(r) \quad (0 \leq r \leq 1). \quad (2.2)$$

Axisymmetric contact problems for rough elastic bodies [6] are reduced to the same integral equation.

The following static condition must be added to Eq. (2.2)

$$Q = 2\pi \int_0^1 r \varphi(r) dr, \quad (2.3)$$

expressing the equilibrium of the die on the surface of the coating.

Let us investigate the properties of the kernel $K(z, \xi)$ (2.1). At all $0 \leq z < \infty$, $0 \leq \xi < \infty$, we will have the following representation for $K(z, \xi)$

$$K(z, \xi) = \frac{2}{\pi(z + \xi)} K\left(\frac{2\sqrt{z\xi}}{z + \xi}\right) - F(\xi, z),$$

where $K(e)$ is a complete elliptic integral of the first kind and $F(\xi, z)$ is a function which is continuous with all derivatives with respect to the set of all variables ξ, z in the circle $0 \leq \xi^2 + z^2 \leq R^2 < \infty$. Allowing for this, we can derive the lemma.

Lemma 1. The integral operator

$$B\varphi = \int_0^1 \rho \varphi(\rho) K\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right) d\rho$$

acts on $L_2(\Omega)$ in $L_2(\Omega)$ continuously, where Ω is a circle $r \leq 1$. Thus, integral equation (2.2) is a Fredholm equation of the second kind. Then it follows from the general theory of such integral equations [3] that (2.2) is unambiguously solvable in $L_2(\Omega)$ with nearly all values of the parameters $c, \lambda \in (0, \infty)$ if $f(r) \in L_2(\Omega)$. If $f(r) \in C(\Omega)$, then $\varphi(r) \in C(\Omega)$ also and, by virtue of the uniqueness of the solution of Eq. (2.2), it can be proven that a solution of the integral equation belonging to $C(\Omega)$ is at the same time a solution of this equation in $L_2(\Omega)$, and vice versa. Theorem 1 follows from this.

Theorem 1. If $f(r) \in C(\Omega)$, then a solution of integral equation (2.2) exists in the space $L_2(\Omega)$ (or $C(\Omega)$) and is unique for any values of the parameters $c, \lambda \in (0, \infty)$.

For proof, allowing for the fact that $\varphi(r) \equiv 0$ at $r > 1$, we perform a scalar multiplication of (2.2) by $r\varphi(r) \in C(\Omega)$. We obtain

$$\int_0^1 r \varphi^2(r) dr + c \int_0^\infty \Phi^2(u) L(u) du = \int_0^1 r f(r) \varphi(r) dr; \quad (2.4)$$

$$\Phi(u) = \int_0^1 r \varphi(r) J_0\left(u \frac{r}{\lambda}\right) dr. \quad (2.5)$$

Equation (2.4) can be written by virtue of the asymptotic formulas

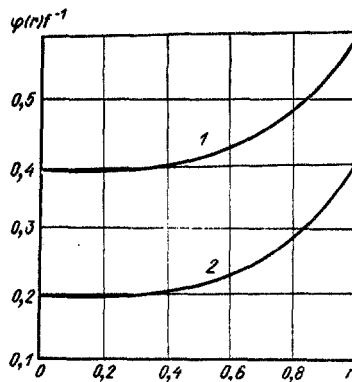


Fig. 3

$$L(u) = Au + O(u^3) \quad (u \rightarrow 0); \quad L(u) = 1 + O(e^{-2u}) \quad (u \rightarrow \infty);$$

$$A = 0.5; \quad A = 4(\kappa - 1)(\kappa + 1)^2$$

and the estimates

$$\Phi(u) = O(u^{-3/2}) \quad (u \rightarrow \infty),$$

obtained from (2.5) using the integral [2]

$$\int_0^1 r J_0(ar) dr = \frac{1}{a} J_1(a).$$

If $f(z) \equiv 0$, $r \in \Omega$, then $\Phi(r) \equiv 0$ ($0 \leq r < \infty$), since $L(u) > 0$ ($0 \leq u < \infty$), and the theorem is proven.

Let us proceed to the construction of an approximate solution of integral equation (2.2). For this, we represent the function $\varphi(r)$ in the form of the following series in a system of polynomials orthonormalized on the interval $[0, 1]$ with the weight r :

$$\varphi(r) = \sum_{m=0}^{\infty} a_m P_{2m}^*(r); \quad P_{2m}^*(r) = 2 \sqrt{m + \frac{1}{2}} P_m(1 - 2r^2). \quad (2.6)$$

Here, $P_m(x)$ are Legendre polynomials.

In accordance with Theorem 1, series (2.6) converges in the norm of the space $L_2(\Omega)$, and $\{a_m\} \in \mathcal{L}_2$ by virtue of Parseval's equation [3] ($\mathcal{L}_2$ is the complete space of quadratically summable sequences). We expand the kernel $K(\rho\lambda^{-1}, r\lambda^{-1})$ and the function $f(r)$ into double and single series, respectively, in the above system of polynomials. We will have

$$K\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}(\lambda) P_{2i}^*(\rho) P_{2j}^*(r); \quad (2.7)$$

$$c_{ij}(\lambda) = \int_0^{\infty} L(u) R_i\left(\frac{u}{\lambda}\right) R_j\left(\frac{u}{\lambda}\right) du;$$

$$R_k(z) = \int_0^1 r J_0(rz) P_{2k}^*(r) dr;$$

$$f(r) = \sum_{m=0}^{\infty} f_m P_{2m}^*(r), \quad f_m = \int_0^1 r f(r) P_{2m}^*(r) dr. \quad (2.8)$$

Using the properties of $K(\rho\lambda^{-1}, r\lambda^{-1})$ indicated above and the limitations placed on the function $f(r)$ in Theorem 1, we can conclude that series (2.7), (2.8) converge in the norm of the space $L_2(\Omega)$ and that the sequences of their coefficients belong to the space $\mathcal{L}_2$, i.e.

TABLE 1

λ	$c_{00}(\lambda)$	$c_{01}(\lambda)$	$c_{02}(\lambda)$	$c_{11}(\lambda)$	$c_{12}(\lambda)$	$c_{22}(\lambda)$
2	2,1574	0,1848	-0,0221	1,4353	0,1047	1,2568
4	3,8228	0,3898	-0,0448	1,8712	0,2108	1,5130

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}^2 < \infty; \quad \sum_{i=0}^{\infty} f_i^2 < \infty. \quad (2.9)$$

It can be proven that with large λ

$$c_{ij}(\lambda) \sim O(\lambda^{-2(i+j)}).$$

Lemma 2. If the function $f(r) \in C(\Omega)$, then any solution $\varphi(r)$ of Eq. (2.2) from the class $C(\Omega)$ corresponds to a sequence of numbers $\{a_m\}$ from the class $\mathcal{L}_2$, satisfying an infinite system of linear algebraic equations

$$a_m = -c \sum_{n=0}^{\infty} c_{mn}(\lambda) a_n + f_m \quad (m = 0, 1, \dots). \quad (2.10)$$

Conversely, if the function $f(r) \in C(\Omega)$, then any solution $\{a_m\}$ from the class $\mathcal{L}_2$ of system (2.10) corresponds to a solution $\varphi(r) \in C(\Omega)$ of Eq. (2.2) of type (2.6).

To prove this, with allowance for Theorem 1, we insert the functions $\varphi(r)$, $f(r)$, $K(\rho\lambda^{-1}, r\lambda^{-1})$ in the form (2.6)-(2.8) into Eq. (2.2). We obtain a relation whose left and right parts consist of series in Legendre polynomials. Equating the coefficients in both parts with polynomials of the same number, we obtain infinite system (2.10).

Now let the sequence $\{a_m\}$ be the solution of infinite algebraic system (2.10) and belong to $\mathcal{L}_2$. Then series (2.6) converges in the norm of the space $L_2(\Omega)$ at $0 \leq r \leq 1$ to a certain function $\varphi(r)$ (here, the Parseval equation should be kept in mind [3]). By an inverse transformation of infinite system (2.10) to integral equation (2.2), it can be made certain that function $\varphi(r) \in L_2(\Omega)$ is a solution of (2.2). Then, by virtue of the considerations presented above, $\varphi(r) \in C(\Omega)$ and the lemma is proven.

Theorem 2. If the function $f(r) \in C(\Omega)$, then an operator consisting of the right part of (2.10) operates from the space $\mathcal{L}_2$ into $\mathcal{L}_2$ continuously with all values of the parameters c , $\lambda \in (0, \infty)$. System (2.10) is unambiguously solvable in $\mathcal{L}_2$.

The first proof of the theorem follows directly from (2.9). Then Hilbert's alternative on the solvability of infinite systems [3] is applicable to system (2.10). Since a solution of integral equation (2.2) corresponding to infinite system (2.10) exists and is unique in the space $C(\Omega)$, then by virtue of the lemma there exists a unique nontrivial solution of system (2.10) belonging to the space $\mathcal{L}_2$. This solution can be found with any degree of accuracy either by the method of reduction or by the method of successive approximations.

Let us illustrate the above with a numerical example concerning a problem of the indentation of the surface of a layer, lying without friction on a rigid base and having its top boundary reinforced by a rod layer, by a rigid die with a flat base ($\delta(r) \equiv \delta \equiv \text{const}$) of radius a exerting a force P . We will set $c = 1$ in (2.2). Table 1 shows values of the coefficients of the expansion $c_{ij}(\lambda)$ with $\lambda = 2$ and 4. Figure 3 shows the distributions of the contact stresses ($\lambda = 2 - (1)$, $\lambda = 4 - (2)$). Here, the quantity $Qf^{-1} = \pi\sqrt{2}a_0$ is equal to 1.473 and 0.841 with $\lambda = 2, 4$. The error of the results shown does not exceed 3%. Here, it was sufficient to take no more than three equations in abbreviated system (2.10).

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SELF-SIMILAR PROBLEM OF INDENTATION OF A PLANE STAMP INTO AN ELASTIC HALFSPACE

O. G. Goman

UDC 539.3.01

One solves the self-similar problem on the indentation of a wedge-shaped stamp into an elastic halfspace in the case when the point of contact of the stamp with the free surface of the halfspace moves with a velocity larger than the velocity of the longitudinal waves. In the case when the velocity of the indicated point is less than the velocity of the Rayleigh waves, the problem has been solved in [1].

1. Formulation of the Problem. Assume that a rigid infinitely long wedge with semi-opening angle β , close to $\pi/2$, is indented into an elastic halfspace with a constant velocity D and assume that the velocity of the displacement of the point of contact with the free surface of the halfspace is $c = D \tan \beta > a$, where a is the velocity of the longitudinal waves. One assumes that the halfspace is under the conditions of plane strain and that outside the contact area, the boundary is free of stresses and there is no friction between the wedge and the medium. One has null initial conditions.

We take the equations of the theory of elasticity in the form needed in the sequel:

$$\frac{\partial v_x}{\partial t} = \frac{1}{\rho} \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right); \quad (1.1)$$

$$\frac{\partial \tau_{xy}}{\partial t} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right), \quad (1.2)$$

where v_x and v_y are the velocity components.

We represent each unknown function f as a sum of two components: f^1 due to the longitudinal waves and f^2 due to the transversal waves [2]. In this case, for the longitudinal component we have

$$\text{rot } \vec{v}^1 = 0, \quad (1.3)$$

and for the transversal component we have

$$\text{div } \vec{v}^2 = 0. \quad (1.4)$$

As it is known, the components f^1 and f^2 satisfy, respectively, the equations of the longitudinal waves

$$\frac{\partial^2 f^1}{\partial x^2} + \frac{\partial^2 f^1}{\partial y^2} = \frac{1}{a^2} \frac{\partial^2 f^1}{\partial t^2} \quad (1.5)$$

and of the transversal waves

$$\frac{\partial^2 f^2}{\partial x^2} + \frac{\partial^2 f^2}{\partial y^2} = \frac{1}{b^2} \frac{\partial^2 f^2}{\partial t^2}. \quad (1.6)$$

The problem under consideration is self-similar and in the self-similar variables $x_0 = x/at$, $y_0 = y/at$, Eqs. (1.1) and (1.2) will have the form (the subscripts 0 will be omitted)

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