

Integral formulation for a circular cylindrical cavity in infinite solid and a finite length coaxial cylindrical crack compressed axially

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Abstract

A method for solving problems of fracture of an infinite solid with a circular cylindrical cavity and a coaxial cylindrical crack near the surface under an uniform axial compression is proposed using a non-classical criterial approach associated with a mechanism of a local stability loss near the defect. The theory of integral Fourier transforms and series expansions are used to reduce these problems to a system of paired integral equations and then to a system of linear algebraic equations with respect to the contraction parameter.

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1. Introduction

Elastic bodies with cracks under compression normal to the surface represent a special class of problems because they are not considered within the scope of the traditional linear fracture mechanics discipline. When the forces are applied along the crack the stress intensity factors, crack opening displacements and energy release rate calculated on the basis of the linear fracture mechanics approach are independent of the loads [1]. Hence, the fracture criterion of Griffith–Irwin [2,3] is not applicable to this scheme of loading. Experiments have been done to confirm what has been said above. A general approach to the study of the initial stage of fracture of solids subjected to the loadings directed along defects has been developed in [4]. According to this criterion the fracture process is assumed to correspond with the local stability loss near the crack and quantified in terms of a critical eigenvalue. The local instability of the materials is examined on the basis of relations of the three-dimensional linearized mechanics of deformed bodies [5,6]. Symmetric and non-axisymmetric

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failure of the half-space with cracks in compression were analyzed in [7,8]. Fracture problems for an infinite solid with a circular cylindrical crack under compression have been discussed previously [9].

This paper considers a method for studying the initial phase of failure of a solid with a cylindrical cavity weakened with a coaxial subsurface cylindrical crack in uniform axial compression along the defect. The analytical solution has been carried out in common form for different models of materials (elastic, composite), the theory of large or small initial strains. The model of solid has to be specified on the final stage of the examination, i.e., the numerical solution. Some problems of failure of materials have been considered in [10,11].

2. Basic equations

A common approach is used to solve compressible and incompressible elastic bodies with an arbitrary elastic potential, both for large subcritical deformations and for two variants of theory of small subcritical deformations. The treatment is limited to the case of unequal roots of the characteristic equation [5,6]. In formulating the linearized problem, use is made of the Lagrangian coordinates x_j that coincide with Cartesian coordinates in the undeformed state, as well as perturbations in the displacement vector u_j and stress tensor t relative to unit area of the body in the undeformed state.

Assume that a infinite solid with a circular cylindrical cavity is subjected to a uniform uniaxial compression along the x_3 axis. The initial forces applied at infinity create a homogeneous stressed-deformed subcritical state in the neighborhood of a defect, which is determined by the following relationships [12]

$$S_{11}^0 = S_{22}^0 = 0, \quad S_{33}^0 = \text{const} \neq 0, \quad u_j^0 = \delta_{jm}(\lambda_j - 1)x_m, \quad \lambda_j = \text{const}, \quad \lambda_1 = \lambda_2 \neq \lambda_3, \quad (\lambda_3 < 1). \quad (1)$$

Here S_{ij}^0 are the components of the symmetric stress tensor relative to a unit area of the body in the undeformed state, u_j^0 are the displacements corresponding to the initial stresses S_{ij}^0 , λ_j ($j = 1, 2, 3$) are the contractions along the axes, and δ_{ij} is the Kronecker symbol. A superscript “0” in Eq. (1) and in the following indicates quantities pertaining to the initial state. The corresponding perturbed quantities are indicated without any additional indices.

The linearized equilibrium equations in the underformed state coordinates are [12]

$$L_{m\alpha}u_\alpha = 0, \quad L_{m\alpha} = \omega_{im\alpha\beta} \frac{\partial^2}{\partial x_i \partial x_\beta}, \quad (2)$$

for compressible bodies and

$$L_{m\alpha}u_\alpha + q_{\alpha m} \frac{\partial u_\alpha}{\partial x_\alpha} = 0, \quad q_{m\alpha} \frac{\partial u_\alpha}{\partial x_m} = 0, \quad L_{m\alpha} = \varkappa_{im\alpha\beta} \frac{\partial^2}{\partial x_i \partial x_\beta}, \quad u_4 \equiv p, \quad (3)$$

for incompressible bodies.

The components of the Green deformation tensor ε^0 in the theory of large initial deformations are related to the contractions λ_j by [12]

$$2\varepsilon_{ij}^0 = \delta_{ij}(\lambda_j^2 - 1), \quad (4)$$

while in the theory of small initial deformations are related to the contractions λ_j by

$$\varepsilon_{ij}^0 = \delta_{ij}(\lambda_j - 1). \quad (5)$$

In the case of an uniform subcritical state (1) the linearized equilibrium equations (2) for compressible solids have the form [12]

$$\omega_{ij\alpha\beta} \frac{\partial^2 u_\alpha}{\partial x_i \partial x_\beta} = 0. \quad (6)$$

Here the components of the tensor ω for the theory of large initial deformations are given by [12]

$$\omega_{ij\alpha\beta} = \lambda_\alpha \lambda_j [\delta_{ij} \delta_{\alpha\beta} A_{i\beta} + (1 - \delta_{ij})(\delta_{i\alpha} \delta_{j\beta} + \delta_{i\beta} \delta_{j\alpha}) \mu_{ij}] + \delta_{i\beta} \delta_{j\alpha} S_{\beta\beta}^0, \quad (7)$$

while for the theory of small initial deformations as

$$\omega_{ij\alpha\beta} = \delta_{ij}\delta_{\alpha\beta}A_{i\beta} + (1 - \delta_{ij})(\delta_{i\alpha}\delta_{j\beta} + \delta_{i\beta}\delta_{j\alpha})\mu_{ij} + \delta_{j\beta}\delta_{j\alpha}S_{\beta\beta}^0, \quad S_0^{\beta\beta} \equiv \sigma_{\beta\beta}^0. \quad (8)$$

In the case of an uniform subcritical state (1) the linearized equilibrium equations (2) for incompressible solids are [12]

$$\mathfrak{x}_{ij\alpha\beta} \frac{\partial^2 u_\alpha}{\partial x_i \partial x_\beta} + q_{ij} \frac{\partial p}{\partial x_i} = 0, \quad q_{ij} \frac{\partial u_j}{\partial x_i} = 0, \quad (9)$$

where the components of the tensor $\mathfrak{x}$ for the theory of large initial deformations are given by [12]

$$\mathfrak{x}_{ij\alpha\beta} = \lambda_j \lambda_\alpha [\delta_{ij}\delta_{\alpha\beta}A_{i\beta} + (1 - \delta_{ij})(\delta_{i\alpha}\delta_{j\beta} + \delta_{i\beta}\delta_{j\alpha})\mu_{ij}] + \delta_{i\beta}\delta_{j\alpha}S_{\beta\beta}^0, \quad (10)$$

while for the theory of small initial deformations are

$$\mathfrak{x}_{ij\alpha\beta} = \delta_{ij}\delta_{\alpha\beta}A_{i\beta} + (1 - \delta_{ij})(\delta_{i\alpha}\delta_{j\beta} + \delta_{i\beta}\delta_{j\alpha})\mu_{ij} + \delta_{j\beta}\delta_{j\alpha}S_{\beta\beta}^0. \quad (11)$$

The values μ_{ij} , $A_{i\beta}$ and, consequently, components of the tensors ω and $\mathfrak{x}$ are determined accordingly to the type of the elastic potential. For a hyperelastic body, where the elastic potential is given as a function of the principal values of the Green deformation tensor $\Phi_0(\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0)$ in the theory of large initial deformations the values S_{ij}^0 , $A_{i\beta}$, and μ_{ij} for incompressible solids are defined as [12]

$$\begin{aligned} A_{ij} &= \frac{\partial^2 \Phi_0}{\partial \varepsilon_i^0 \partial \varepsilon_j^0} - 2p_0 \delta_{ij} \lambda_j^{-4}, \quad S_{ii}^0 = \frac{\partial \Phi_0}{\partial \varepsilon_i^0} + \lambda_i^{-2} p_0; \\ \mu_{ij} &= \frac{1}{2(\varepsilon_i^0 - \varepsilon_j^0)} \left(\frac{\partial}{\partial \varepsilon_i^0} - \frac{\partial}{\partial \varepsilon_j^0} \right) \Phi_0 - p_0 \lambda_i^{-2} \lambda_j^{-2}, \\ \lambda_1 \lambda_2 \lambda_3 &= 1, \quad q_i = \lambda_i^{-1}. \end{aligned} \quad (12)$$

The expressions for the theory of small initial deformations are as follows

$$\begin{aligned} A_{ij} &= \frac{\partial^2 \Phi_0}{\partial \varepsilon_i^0 \partial \varepsilon_j^0}, \quad \sigma_{\beta\beta}^0 = \frac{\partial \Phi_0}{\partial \varepsilon_i^0} + p_0; \\ \mu_{ij} &= \frac{1}{2(\varepsilon_i^0 - \varepsilon_j^0)} \left(\frac{\partial}{\partial \varepsilon_i^0} - \frac{\partial}{\partial \varepsilon_j^0} \right) \Phi_0, \quad \varepsilon_i^0 = \lambda_i - 1. \end{aligned} \quad (13)$$

For a hyperelastic body, where the elastic potential is given as a function of the principal values of the Green deformation tensor $\Phi_0(\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0)$ in the theory of large initial deformations S_{ij}^0 , $A_{i\beta}$, and μ_{ij} for compressible solids have the form [12]

$$A_{ij} = \frac{\partial^2 \Phi_0}{\partial \varepsilon_i^0 \partial \varepsilon_j^0}, \quad S_{ii}^0 = \frac{\partial \Phi_0}{\partial \varepsilon_i^0}; \quad \mu_{ij} = \frac{1}{2(\varepsilon_i^0 - \varepsilon_j^0)} \left(\frac{\partial}{\partial \varepsilon_i^0} - \frac{\partial}{\partial \varepsilon_j^0} \right) \Phi_0. \quad (14)$$

The system of linearized equilibrium equations (6) and (9) for compressible and incompressible solids, respectively, consists of partial differential equations with constant coefficients and in the case of uniform subcritical deformations, its general solutions can be constructed in terms of potential functions. Let the coordinate system x_m is now chosen to be a cylindrical polar system (r, θ, x_3) with the axis x_3 as one of the cylindrical cavity. The general solutions of the static linearized equations (6) and (9) for axially symmetric problems in the case of unequal roots $n_1^0 \neq n_2^0$ of the governing characteristic equation are given in terms of two potential functions $\varphi_i(r, x_3) (i = 1, 2)$ which obey the equations

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + n_i \frac{\partial^2}{\partial x_3^2} \right) \varphi_i(r, x_3) = 0 \quad (i = 1, 2), \quad (15)$$

and have the following form [12]

$$u_r = \frac{\partial \varphi_1}{\partial r} + \frac{\partial \varphi_2}{\partial r}; \quad u_3 = m_1 \frac{\partial \varphi_1}{\partial r} + m_2 \frac{\partial \varphi_2}{\partial r}. \quad (16)$$

The representations for the components of the Kirchhoff stress tensor t on the cylindrical surface are given by [9]

$$\begin{aligned} t_{r3} &= C_{44} \left[d_1 \frac{\partial^2 \varphi_1}{\partial r \partial x_3} + d_2 \frac{\partial^2 \varphi_2}{\partial r \partial x_3} \right]; \\ t_{rr} &= C_{44} \left[-p_1 \left(l_1 \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial x_3^2} \right) \varphi_1 - p_2 \left(l_2 \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial x_3^2} \right) \varphi_2 \right]. \end{aligned} \quad (17)$$

The quantities in Eqs. (16) and (17) depend on the choice of material (form of the elastic potential) and the initial stress-deformed state.

For compressible solids, they can be written in terms of the components of the tensor ω as

$$\begin{aligned} C_{44} &= \omega_{1313}, \quad m_i = (\omega_{1111} n_i - \omega_{3113})(\omega_{1133} + \omega_{1313})^{-1}, \quad d_i = 1 + m_i \omega_{1331} \omega_{1313}^{-1}, \\ p_i &= (\omega_{1111} n_i + \omega_{1133} m_i) \omega_{1313}^{-1}, \quad l_j = (\omega_{1111} - \omega_{1122})(\omega_{1111} n_i + \omega_{1133} m_i)^{-1} \quad (i = 1, 2). \end{aligned} \quad (18)$$

For incompressible solids, they are given in terms of the components of the tensor $\varkappa$ as follows

$$\begin{aligned} C_{44} &= \varkappa_{1313}, \quad m_j = q_{11} q_{33}^{-1} n_j, \quad d_j = 1 + \varkappa_{1331} \varkappa_{1313}^{-1} m_j; \\ p_j &= [q_{11} q_{33}^{-1} (\varkappa_{1133} - \varkappa_{1313}) n_j - \varkappa_{1133} m_j + \varkappa_{3113}] \varkappa_{1313}^{-1}, \\ l_j &= (\varkappa_{1111} - \varkappa_{1122}) [q_{11} q_{33}^{-1} (\varkappa_{1133} - \varkappa_{1313}) n_j - \varkappa_{1133} m_j + \varkappa_{3113}]^{-1} \quad (j = 1, 2), \end{aligned} \quad (19)$$

where the roots n_1 and n_2 of the governing characteristic equation are for compressible solids

$$\begin{aligned} n_{1,2} &= c \pm \sqrt{c^2 - \omega_{3333} \omega_{3113} \omega_{1111}^{-1} \omega_{1331}^{-1}} \\ 2\omega_{1111} \omega_{1331} c &= \omega_{1111} \omega_{3333} + \omega_{1331} \omega_{3113} - (\omega_{1133} + \omega_{1313})^2, \end{aligned} \quad (20)$$

and for incompressible solids

$$\begin{aligned} n_{1,2} &= c \pm \sqrt{c^2 - q_{33}^2 q_{11}^{-2} \varkappa_{3113} \varkappa_{1331}^{-1}} \\ 2q_{11}^2 \varkappa_{1331} c &= q_{11}^2 \varkappa_{3333} + q_{33}^2 \varkappa_{1111} - 2q_{11} q_{33} (\varkappa_{1133} + \varkappa_{1313}). \end{aligned} \quad (21)$$

3. Problem formulation

Consider an infinite solid containing a circular cylindrical cavity of radius b and a circular cylindrical crack of radius $b+h$ and length $2a$, located on the lateral part of the cylindrical surface $\{r = b+h, 0 \leq \theta < 2\pi, -a \leq x_3 \leq a\}$, whose axis coincides with the cavity axis x_3 and which is compressed along the x_3 axis by uniform forces applied at infinity. Here (r, θ, x_3) be the cylindrical coordinates and h is the distance between the surface of the cylindrical cavity and the surface of the circular cylindrical crack. As a result of the compression parallel to the crack axis, a homogeneous initial stress and strain state (1) occurs near the crack-like defect. Separate the space into two regions $b < r < b+h$ and $b+h < r < \infty$. Denote by the superscript 1 the quantities referred to the region $b < r < b+h$ and by the superscript 2 the ones referred to the region $b+h < r < \infty$. Assume that the edges of the crack, as well as the surface of the cavity, are free of stresses. Here it assumed that the surfaces of the defect do not come into a contact interaction. In addition, at the boundary of regions 1 ($b < r < b+h$) and 2 ($b+h < r < \infty$) the continuity conditions for the displacements and stresses must be satisfied outside the crack. Given these remarks, we write the boundary conditions for the linearized problem in the following form:

$$t_{rr}^{(1)} = 0, \quad t_{r3}^{(1)} = 0, \quad \text{when } \rho = \rho_0, -\infty \leq \varsigma \leq +\infty; \quad (22)$$

$$t_{rr}^{(1)} = t_{rr}^{(2)} = 0, \quad t_{r3}^{(1)} = t_{r3}^{(2)} = 0, \quad \text{when } \rho = 1, -\beta \leq \varsigma \leq \beta; \quad (23)$$

$$u_r^{(1)} = u_r^{(2)}, \quad u_3^{(1)} = u_3^{(2)}, \quad \text{when } \rho = 1, |\varsigma| > \beta; \quad (24)$$

$$t_{rr}^{(1)} = t_{rr}^{(2)}, \quad t_{r3}^{(1)} = t_{r3}^{(2)}, \quad \text{when } \rho = 1, 0 \leq |\varsigma| < +\infty. \quad (25)$$

where $\rho = r/(b+h)$; $\rho_0 = b/(b+h)$; $\varsigma = x_3/(b+h)$ and $\beta = a/(b+h)$ are dimensionless quantities and the superscripts 1 and 2 refer to regions 1 and 2, respectively. Thus, the boundary conditions include conditions at the surface of the cavity (22), conditions at the edges of the cracks (23), and continuity conditions for the displacements (24) and stresses (25) at the interface between the regions 1 ($b < r < b+h$) and 2 ($b+h < r < \infty$).

4. Derivation of dual integral equations

To obtain a system of dual integral equations, the theory of the Fourier integral transforms is used. Represent the potential functions $\varphi_1(r, x_3)$ and $\varphi_2(r, x_3)$ separately in each of regions 1 ($b < r < b+h$) and 2 ($b+h < r < \infty$) applying the subscripts 1 and 2 to them in the corresponding regions, in the form of Fourier integral transforms

$$C_{44}\varphi_i^{(1)} = \int_0^\infty f_i(\lambda) \cdot I_0(\sqrt{n_i}\lambda\rho) \cdot \cos \varsigma\lambda d\lambda + \int_0^\infty e_i(\lambda) \cdot K_0(\sqrt{n_i}\lambda\rho) \cdot \cos \varsigma\lambda d\lambda, \quad (i = 1, 2) \quad (26)$$

$$C_{44}\varphi_i^{(2)} = \int_0^\infty g_i(\lambda) \cdot K_0(\sqrt{n_i}\lambda\rho) \cdot \cos \varsigma\lambda d\lambda, \quad (i = 1, 2), \quad (27)$$

where $f_i(\lambda)$, $e_i(\lambda)$ and $g_i(\lambda)$ ($i = 1, 2$) are unknown weighting functions; I_0 and K_0 are the modified Bessel functions. When there is no cylindrical cavity in the infinite solids, it is necessary to set $e_i(\lambda) \equiv 0$ in the integral representations [9]. Using expansion, the components of displacements (16) and stresses (17) for the region 2 are obtained:

$$\begin{aligned} C_{44}(b+h)u_r^{(2)} &= - \sum_{i=1}^2 \int_0^{+\infty} g_i(\lambda) \sqrt{n_i} \lambda K_1(\sqrt{n_i}\lambda\rho) \cos \varsigma\lambda d\lambda, \\ C_{44}(b+h)u_3^{(2)} &= - \sum_{i=1}^2 \int_0^{+\infty} g_i(\lambda) m_i \lambda K_0(\sqrt{n_i}\lambda\rho) \sin \varsigma\lambda d\lambda, \\ (b+h)^2 t_{r3}^{(2)} &= - \sum_{i=1}^2 \int_0^{+\infty} g_i(\lambda) d_i \sqrt{n_i} \lambda^2 K_1(\sqrt{n_i}\lambda\rho) \sin \varsigma\lambda d\lambda, \\ (b+h)^2 t_{rr}^{(2)} &= \sum_{i=1}^2 \int_0^{+\infty} g_i(\lambda) p_i \sqrt{n_i} \lambda \left[\frac{\lambda}{\sqrt{n_i}} K_0(\sqrt{n_i}\lambda\rho) + \frac{l_i}{\rho} K_1(\sqrt{n_i}\lambda\rho) \right] \cos \varsigma\lambda d\lambda. \end{aligned} \quad (28)$$

Similarly, for the region 1 the components of displacements (16) and stresses (17) are

$$\begin{aligned} C_{44}(b+h)u_r^{(1)} &= \sum_{i=1}^2 \int_0^{+\infty} \{f_i(\lambda) \sqrt{n_i} \lambda I_1(\sqrt{n_i}\lambda\rho) - e_i(\lambda) \sqrt{n_i} \lambda K_1(\sqrt{n_i}\lambda\rho)\} \cos \varsigma\lambda d\lambda, \\ C_{44}(b+h)u_3^{(1)} &= - \sum_{i=1}^2 \int_0^{+\infty} \{f_i(\lambda) m_i \lambda I_0(\sqrt{n_i}\lambda\rho) + e_i(\lambda) m_i \lambda K_0(\sqrt{n_i}\lambda\rho)\} \sin \varsigma\lambda d\lambda, \\ (b+h)^2 t_{r3}^{(1)} &= - \sum_{i=1}^2 \int_0^{+\infty} \{f_i(\lambda) d_i \sqrt{n_i} \lambda^2 I_1(\sqrt{n_i}\lambda\rho) - e_i(\lambda) d_i \sqrt{n_i} \lambda^2 K_1(\sqrt{n_i}\lambda\rho)\} \sin \varsigma\lambda d\lambda, \\ (b+h)^2 t_{rr}^{(1)} &= \sum_{i=1}^2 \int_0^{+\infty} f_i(\lambda) p_i \sqrt{n_i} \lambda \left\{ \frac{\lambda}{\sqrt{n_i}} I_0(\sqrt{n_i}\lambda\rho) - \frac{l_i}{\rho} I_1(\sqrt{n_i}\lambda\rho) \right\} \\ &\quad + e_i(\lambda) p_i \sqrt{n_i} \lambda \left[\frac{\lambda}{\sqrt{n_i}} K_0(\sqrt{n_i}\lambda\rho) + \frac{l_i}{\rho} K_1(\sqrt{n_i}\lambda\rho) \right] \cos \varsigma\lambda d\lambda. \end{aligned} \quad (29)$$

Satisfying boundary conditions (22)–(25) and taking into account Eqs. (28) and (29), we obtain the following system of the paired integral equations with respect to the new unknown functions $E_r(\lambda)$ and $E_3(\lambda)$

$$\int_0^{+\infty} \lambda \left\{ [\alpha_{11} + \overline{\psi}_{11}(\lambda)] E_r(\lambda) + \overline{\psi}_{12}(\lambda) \cdot E_3(\lambda) \right\} \cos \varsigma \lambda d\lambda = 0 \quad (0 \leq \varsigma < \beta);$$

$$\int_0^{+\infty} \lambda \left\{ [\alpha_{22} + \overline{\psi}_{22}(\lambda)] E_3(\lambda) + \overline{\psi}_{21}(\lambda) \cdot E_r(\lambda) \right\} \sin \varsigma \lambda d\lambda = 0 \quad (0 \leq \varsigma < \beta);$$

and

$$\int_0^{+\infty} E_r(\lambda) \cos \varsigma \lambda d\lambda = 0 \quad (\beta < \varsigma < \infty);$$

$$\int_0^{+\infty} E_3(\lambda) \sin \varsigma \lambda d\lambda = 0 \quad (\beta < \varsigma < \infty).$$

Here

$$\overline{\psi}_{11}(\lambda) = \overline{\psi}_{22}(\lambda) = O(1/\lambda^2), \quad \overline{\psi}_{12}(\lambda) = \overline{\psi}_{21}(\lambda) = O(1/\lambda),$$

$E_r(\lambda)$ and $E_3(\lambda)$ are unknown functions; α_{11} and α_{22} are constants. The obtained paired integral equations are reduced to a system of linear algebraic equations by use of the series expansion method.

5. Solution of the system of paired integral equations

In solving the system of paired integral equations, we shall choose a solution in a form such that the integral Eq. (31) specified over the interval $\beta < \varsigma < \infty$ are satisfied identically. The remaining equations are reduced to a system of algebraic equations. To do this we expand the unknown functions $E_r(\lambda)$ and $E_3(\lambda)$ into infinite series of Bessel functions of the first kind J_i and with the unknown coefficients a_j and b_j [13]

$$E_r(\lambda) = \sum_{j=0}^{\infty} a_j \lambda^{-1} J_{2j+1}(\beta \lambda), \quad E_3(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^{-1} J_{2j+2}(\beta \lambda).$$

Direct substitution of expansions (33) into (31) causes them to be satisfied identically [13], while (30) take the form

$$\int_0^{+\infty} \left\{ \sum_{j=0}^{+\infty} a_j [\alpha_{11} + \overline{\psi}_{11}(\lambda)] J_{2j+1}(\beta \lambda) + \sum_{j=0}^{+\infty} b_j \overline{\psi}_{12}(\lambda) J_{2j+2}(\beta \lambda) \right\} \cos \varsigma \lambda d\lambda = 0,$$

$$\int_0^{+\infty} \left\{ \sum_{j=0}^{+\infty} a_j \overline{\psi}_{21}(\lambda) J_{2j+1}(\beta \lambda) + \sum_{j=0}^{+\infty} b_j [\alpha_{22} + \overline{\psi}_{22}(\lambda)] J_{2j+2}(\beta \lambda) \right\} \sin \varsigma \lambda d\lambda = 0.$$

Using the representations of the functions $\cos \varsigma \lambda$ and $\sin \varsigma \lambda$ in the form of series as follows [13]

$$\cos \varsigma \lambda = \sum_{i=0}^{\infty} \varepsilon_i J_{2i}(\beta \lambda) \cos 2i\varphi; \quad \varepsilon_0 = 1, \quad \varepsilon_i = 2, \quad (i \geq 1);$$

$$\sin \varsigma \lambda = \sum_{i=0}^{\infty} 2J_{2i+1}(\beta \lambda) \sin(2i+1)\varphi; \quad \varphi = \arcsin(\varsigma/\beta),$$

and equating the coefficients of the harmonics in Eq. (34) to zero, the following homogeneous system of linear algebraic equations with respect to the unknown a_j and b_j are found:

$$\sum_{j=0}^{+\infty} a_j [\alpha_{11} P_{ij}^{(0)}(\beta \lambda) + P_{ij}(\beta \lambda)] + \sum_{j=0}^{+\infty} b_j Q_{ij}(\beta \lambda) = 0, \quad i = 0, 1, 2, \dots;$$

$$\sum_{j=0}^{+\infty} a_j R_{ij}(\beta \lambda) + \sum_{j=0}^{+\infty} b_j [\alpha_{22} T_{ij}^{(0)}(\beta \lambda) + T_{ij}(\beta \lambda)] = 0, \quad i = 0, 1, 2, \dots,$$

where the values $P_{ij}(\beta \lambda)$, $Q_{ij}(\beta \lambda)$, $R_{ij}(\beta \lambda)$ and $T_{ij}(\beta \lambda)$ denote the following expression:

$$\begin{aligned}
P_{ij}(\beta\lambda) &= \int_0^{+\infty} \overline{\psi}_{11}(\lambda) J_{2i}(\beta\lambda) J_{2j+1}(\beta\lambda) d\lambda; \\
R_{ij}(\beta\lambda) &= \int_0^{+\infty} \overline{\psi}_{21}(\lambda) J_{2i+1}(\beta\lambda) J_{2j+1}(\beta\lambda) d\lambda; \\
T_{ij}(\beta\lambda) &= \int_0^{+\infty} \overline{\psi}_{22}(\lambda) J_{2i+1}(\beta\lambda) J_{2j+2}(\beta\lambda) d\lambda; \\
Q_{ij}(\beta\lambda) &= \int_0^{+\infty} \overline{\psi}_{12}(\lambda) J_{2i}(\beta\lambda) J_{2j+2}(\beta\lambda) d\lambda \quad (i, j = 0, 1, 2, \dots),
\end{aligned} \tag{37}$$

and also $M_{ij}^{(0)}(\beta\lambda)$ and $N_{ij}^{(0)}(\beta\lambda)$, which are reduced to

$$P_{ij}^{(0)}(\beta\lambda) = T_{ij}^{(0)}(\beta\lambda) = \begin{cases} (-1)^{j-i}/2\beta & \text{where } i \leq j; \\ (-1)^{i-j-1}/2\beta & \text{where } i > j. \end{cases} \tag{38}$$

The homogeneous system of algebraic (36) has a non-trivial solution if the determinant of the system equals to zero, i.e for incompressible solids

$$\det \|d_{kl}(\beta, \rho^*, \lambda_3, \mathfrak{a})\| = 0 \quad (k, l = 1, 2, 3, \dots), \tag{39}$$

and compressible solids

$$\det \|d_{kl}(\beta, \rho^*, \lambda_3, \omega)\| = 0 \quad (k, l = 1, 2, 3, \dots). \tag{40}$$

Here the components of the tensors ω and $\mathfrak{a}$ for the theory of large and small initial deformations are defined are determined by Eqs. (7), (8) and (10), (11), respectively. Therefore, we have reduced the linearized problem formulated above to an eigenvalue problem for Eq. (36) in terms of the parameter $\lambda_3 < 1$, which should be solved numerically.

6. Some models of solid

The components of the tensors ω and $\mathfrak{a}$ in Eqs. (39), (40) are determined accordingly to the type of the elastic potential. The elastic potential (Treloar potential) for incompressible Neo-Hookean type solids is given as [12]

$$W^0 = 2C_{10}A_1 = 2C_{10}(\varepsilon_1^0 + \varepsilon_2^0 + \varepsilon_3^0), \tag{41}$$

where C_{10} is a constant of the body; $\varepsilon_i^0 = (\lambda_i^2 - 1)/2$ are the principal values of the Green's deformation tensor. The values μ_{ij} , $A_{i\beta}$ in (12) for the solid with the Treloar potential (41) are

$$A_{i\beta} = -2p_0\delta_{i\beta}\lambda_\beta^{-4}, \quad \mu_{ij} = p_0\lambda_i^{-2}\lambda_j^{-2}; \quad S_{\beta\beta}^0 = 2C_{10} + \lambda_\beta^{-2}p_0, \quad \lambda_1\lambda_2\lambda_3 = 1; \quad q_1 = q_2 \neq q_3. \tag{42}$$

In the case of an uniform uniaxial compression along the x_3 axis we have [12]

$$\lambda_1 = \lambda_2 = \lambda_3^{-1/2}. \tag{43}$$

Combining Eqs. (42), (43), and (10), we obtain

$$\begin{aligned}
\mathfrak{a}_{1111} &= 4C_{10}, \quad \mathfrak{a}_{1122} = \mathfrak{a}_{1133} = 0, \quad \mathfrak{a}_{1313} = 2C_{10}\lambda_3^{-3/2}; \quad \mathfrak{a}_{3113} = \mathfrak{a}_{1331} = 2C_{10}, \\
\mathfrak{a}_{3333} &= 2C_{10}(1 + \lambda_3^{-3}); \quad n_1 = 1, \quad n_2 = \lambda_3^{-3}, \quad m_1 = \lambda_3^{3/2}, \quad m_2 = \lambda_3^{-3/2}; \\
l_1 &= 1, \quad l_2 = 2(1 + \lambda_3^{-3})^{-1}.
\end{aligned} \tag{44}$$

For a compressible body, the elastic potential is given as [12]

$$\Phi = \frac{1}{2}\lambda A_1^2 + \mu A_2, \tag{45}$$

where A_1, A_2 are algebraic invariants of the Green deformation tensor and λ, μ are the Lamé elastic constants, the components of the tensor ω for the theory of small initial deformations in the case of a uniform uniaxial compression along the x_3 axis are

$$\begin{aligned}\omega_{1111} &= \omega_{2222} = \lambda + 2\mu, & \omega_{1221} &= \omega_{1331} = \omega_{2112} = \omega_{2332} = \mu, \\ \omega_{1122} &= \omega_{1133} = \omega_{2211} = \omega_{2233} = \omega_{3311} = \omega_{3322} = \lambda, \\ \omega_{1212} &= \omega_{1313} = \omega_{2121} = \omega_{2323} = \omega_{3131} = \omega_{3232} = \mu, \\ \omega_{3113} &= \mu + \sigma_{33}^0, & \omega_{3223} &= \mu + \sigma_{33}^0, & \omega_{3333} &= \mu + \sigma_{33}^0.\end{aligned}\quad (46)$$

For the case of axially symmetric loading, i.e. $\lambda_1 = \lambda_2 \neq \lambda_3$ the result is

$$\sigma_{33}^0 = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}(\lambda_3 - 1). \quad (47)$$

Substituting Eqs. (46), (47) into Eqs. (18), (20) there renders

$$\begin{aligned}m_1 &= 1 + \frac{\lambda}{\lambda + \mu} \frac{3\lambda + 2\mu}{\lambda + \mu}(\lambda_3 - 1); & m_2 &= 1; \\ d_1 &= 2 + \frac{\lambda}{\lambda + \mu} \frac{3\lambda + 2\mu}{\lambda + \mu}(\lambda_3 - 1); & d_2 &= 2; \\ p_1 &= 2 + \frac{2\lambda + \mu}{\lambda + \mu} \frac{3\lambda + 2\mu}{\lambda + \mu}(\lambda_3 - 1); & p_2 &= 2 + \frac{3\lambda + 2\mu}{\lambda + \mu}(\lambda_3 - 1); \\ l_1 &= \frac{1}{1 + \frac{1}{2} \frac{2\lambda + \mu}{\lambda + \mu} \frac{3\lambda + 2\mu}{\lambda + \mu}(\lambda_3 - 1)}; & l_2 &= \frac{1}{1 + \frac{1}{2} \frac{3\lambda + 2\mu}{\lambda + \mu}(\lambda_3 - 1)}; \\ n_1 &= 1 + \frac{3\lambda + 2\mu}{\lambda + 2\mu}(\lambda_3 - 1); & n_2 &= 1 + \frac{\mu}{\lambda + \mu} \frac{3\lambda + 2\mu}{\lambda + 2\mu}(\lambda_3 - 1).\end{aligned}\quad (48)$$

Described is a method for studying the initial phase of failure of a solid with a cylindrical cavity weakened with a coaxial subsurface cylindrical crack in uniform axial compression along the defect. The analytical solution has been carried out in common form for different models of materials (elastic, composite), the theory of large or small initial strains. The model of solid has to be specified on the final stage of the examination, i.e., the numerical solution. The linearized problem formulated above is reduced to an eigenvalue problem for Eq. (36) in terms of the parameter $\lambda_3 < 1$ in common form for different models of materials, which should be solved numerically.

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