

Stress distribution at sharp and rounded V-notches in quasi-orthotropic plane



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ARTICLE INFO

Article history:

Received 10 September 2015

Revised 1 February 2016

Available online 15 February 2016

Keywords:

Fracture mechanics

Mode I

Sharp and rounded V-notch

Stress intensity factor

Stress concentrations

Singular integral equations

ABSTRACT

The eigensolution of the quasi-orthotropic wedge problem is presented. The degenerate orthotropic material of the wedge is characterized by doubled roots of the characteristic equation. Using the method of singular integral equations, the distribution of the stress field in the quasi-orthotropic plane weakened by semi-infinite rounded V-notch was obtained. The edge of the notch was free of load and the asymptotic stress field defined by a stress intensity factor was set at the infinity. On this basis, the relationship between stress intensity factor at the sharp notch vertex and the normal stress at the vertex of corresponding rounded V-notch with the same opening angle was established. As it was shown in the example, this relationship is asymptotic for finite notched bodies and it can be used to calculate a notch stress intensity factor based on a stress concentration results obtained for a notch rounded with small radius of curvature.

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1. Introduction

Studies of stress and displacement fields in solids with crack-like defects play the key role in fracture mechanics. Such defects include both cracks and sharp V-shaped notches, which in the frames of linear theory of elasticity originate infinite stresses in their tips, as well as rounded (blunted) V-notches with small radii of vertex curvature. In the tips of such notches, even small load generates finite stresses higher than the ultimate strength of material that precludes application of classical criteria for the evaluation of structural members' ability to withstand an applied load.

The limit transition from a rounded notch to sharp V-notch requires the knowledge of the asymptotic relation between the stress intensity and the stress concentration factors for the notch which radius of curvature approaches zero.

This relationship was firstly obtained by Irwin (1958) for parabolic notches, and thereafter it was frequently encountered in later publications (Creager and Paris, 1967; Glinka, 1985; Hasebe and Kutanda, 1978; Neuber, 1977; Sih et al., 1965), also for hyperbolic notches by Atzori and Lazzarin (2000); Benthem (1987); Filippi et al. (2002); Lazzarin and Filippi (2006); Lazzarin and Tovo (1996); Lazzarin et al. (2011); Nui et al. (1994) and Zappalorto et al. (2011). The works of Berto and Lazzarin (2014) and Radaj (2014)

contain an extended overview of previous achievements for isotropic solids. Only recently the results for blunt notches in anisotropic bodies have been reported (Chiang, 1994; Zappalorto and Carraro, 2015a, 2015b).

In our investigations (Savruk and Kazberuk, 2006, 2007, 2010) we have shown that the relationship between stress intensity factor at sharp notch vertex and the stress concentration factor at the vertex of rounded V-notch in isotropic body depend not only on the radius at rounded vertex and the notch opening angle but the shape of the notch in a certain neighborhood from the vertex as well. The difference between the stress concentration factor calculated for hyperbolic notch and the V-notch rounded with circular arc varies depending on the deformation state of an elastic body with V-notch (Savruk and Kazberuk, 2011, 2012a; Savruk et al., 2012). Thus, our approach adopts the simplest and the most unambiguous definition of a rounded V-notch: the notch edges are straight lines connected with circular arc. Comprehensive summary of the results obtained by applying unified approach to the stress concentration at the vertices of sharp and rounded V-notches is comprised in the monograph (Savruk and Kazberuk, 2012b). The aim of this work is to extend the method to the problems of stress concentration in the vicinity of sharp and rounded vertices of V-notches in quasi-orthotropic materials.

The stress distribution in elastic orthotropic sharp wedge has been already studied by Blinowski and Rogaczewski (2000); Bogy (1972); Selvarathinam and Pageau (1997); Wu and Chang (1993); Wu and Liu (2008). We verified and established the eigensolution

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of this problem in the terms of complex potentials and performed limit transition from generally orthotropic to quasi-orthotropic material (Savruk and Kazberuk, 2014). In this paper we present another, direct, approach to the problem of the wedge in quasi-orthotropic plane.

Using the method of auxiliary complex plane proposed by Lekhnitskii (1968) (see also Fil'shtinskiy, 1976 and Ioakimidis and Theocaris, 1977), Savruk and Chornenkyi (2015, 2014b) reduced the problem of the theory of elasticity for quasi-orthotropic plane with curvilinear cracks to an integral equation. The relationships defining singular integral equation are similar to the case of isotropic material. Thus, the numerical solutions of the problems of bodies with cracks, notches and holes can be obtained in exactly the same way as for an isotropic media. The equivalent form of singular equation was obtained on the way of limit transition from the equation for anisotropic body (see Savruk and Kazberuk, 2014).

The quasi-isotropic material (Hasebe and Sato, 2013, 2015; Savruk and Chornenkyi, 2014a, 2014b; Suo, 1990a; Suo et al., 1991) or degenerate orthotropic material is the special class of orthotropic materials typified by double roots of its characteristic equation. This class includes isotropic materials, and various composite materials based on ceramics, fibrous and layered composites, etc. (Suo et al., 1991). In the literature (i.e. Cho et al., 1992) these solids are also called pseudo-isotropic, and in the theory of orthotropic shells – a specially orthotropic (see Erdogan et al., 1974; Kostenko, 1980; Krenk, 1978; Shevchenko et al., 1998).

Here, using the methods of the theory of complex variable we present solution to the theory of elasticity problem for plane quasi-orthotropic wedge under symmetrical load, providing characteristic equations for singularity exponent at the wedge vertex and formulas describing components of singular stress field. Using the singular integral equation method, we found the solution to the problem of infinite rounded V-notch in quasi-orthotropic plane subjected to asymptotic stress field defining notch stress intensity factor (NSIF) at the vertex of the corresponding sharp notch. On this foundation the relationship between NSIF for sharp notch and the stress concentration factor at the vertex of rounded notch was established. Relationships of this type have asymptotic character for finite bodies with notches and they can be used to calculate NSIF based on stress concentration results obtained for notch rounded with small radius of curvature.

2. General equations of the theory of elasticity of an quasi-orthotropic medium

The linear relationships between components of the stress tensor σ_x , σ_y , τ_{xy} and components of the strain tensor ε_x , ε_y , ε_{xy} (generalized Hooke's law) for plane stress state of an orthotropic body have the form (Lekhnitskii, 1968):

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y, \quad \varepsilon_y = a_{12}\sigma_x + a_{22}\sigma_y, \quad 2\varepsilon_{xy} = a_{66}\tau_{xy}, \quad (1)$$

The axes x and y are parallel to the principal axes of orthotropy, and

$$a_{11} = \frac{1}{E_x}, \quad a_{22} = \frac{1}{E_y}, \quad a_{66} = \frac{1}{G}, \quad a_{12} = -\frac{\nu_{xy}}{E_x}.$$

where

$$E_x = E_1, \quad E_y = E_2 \quad (\text{or respectively } E_x = E_2, \quad E_y = E_1),$$

are Young's moduli for directions x and y ;

$$\nu_{xy} = \nu_{12}, \quad \left(\nu_{xy} = \nu_{21} = \nu_{12} \frac{E_2}{E_1} \right)$$

are Poisson's ratios;

$$G = G_{12} = G_{xy} = G_{21} = G_{yx}$$

is a shear modulus.

In the case of plane strain, the elastic constants a_{ij} in Hooke's law (1) are substituted by:

$$a'_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}},$$

where constants a_{i3} describe elastic properties of body at third direction:

$$a_{13} = -\frac{\nu_{13}}{E_1}, \quad a_{23} = -\frac{\nu_{23}}{E_2}, \quad a_{33} = \frac{1}{E_3}$$

We assume that the elastic constants a_{ij} satisfy the relationship

$$a_{66} = 2 \left(\sqrt{a_{11}a_{22}} - a_{12} \right). \quad (2)$$

In the terms of engineering constants, we have

$$G = \frac{E_1}{2} \left(\sqrt{\frac{E_1}{E_2} + \nu_{12}} \right)^{-1}, \quad (3)$$

for plane stress, or

$$G = \frac{1}{2} \left[\sqrt{\left(\frac{1}{E_2} - \frac{\nu_{32}^2}{E_3} \right) \left(\frac{1}{E_1} - \frac{\nu_{31}^2}{E_3} \right)} + \frac{\nu_{12}}{E_1} + \frac{\nu_{31}\nu_{32}}{E_3} \right]^{-1}, \quad (4)$$

for plane strain state respectively (Zhang and Deng, 2008). Therefore, such material is called quasi-orthotropic (Hasebe and Sato, 2013) and the relationships (3) or (4) can be used to determine quasi-orthotropic solid.

We introduce the stress function $F(x, y)$ such that (Lekhnitskii, 1968):

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \quad (5)$$

In the absence of mass forces the stress function $F(x, y)$ for quasi-orthotropic solid satisfies the elliptical differential equation of fourth order:

$$\frac{\partial^4 F}{\partial y^4} + 2\gamma^2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \gamma^4 \frac{\partial^4 F}{\partial x^4} = 0, \quad (6)$$

The characteristic equation of (6) has the following form:

$$\mu^4 + 2\gamma^2 \mu^2 + \gamma^4 = 0 \quad (7)$$

where $\gamma = \sqrt[4]{a_{22}/a_{11}}$ is an orthotropy parameter. In the terms of engineering constants it is equal

$$\gamma = \sqrt[4]{E_x/E_y}$$

for plane stress state. For plane strain, we have (Zhang and Deng, 2008)

$$\gamma = \sqrt[4]{\frac{1/E_y - \nu_{3y}^2/E_3}{1/E_x - \nu_{3x}^2/E_3}},$$

respectively.

Eq. (7) has conjugate multiple roots $\mu_1 = \mu_2 = i\gamma$ and $\bar{\mu}_1 = \bar{\mu}_2 = -i\gamma$. When the parameter $\gamma = 1$ we come to an isotropic material. The general solution of Eq. (6) for quasi-orthotropic body can be expressed by two analytic functions $\varphi_1(z_1)$ and $\chi_1(z_1)$ of the same complex argument $z_1 = x + i\gamma y$ in the form (Lekhnitskii, 1968):

$$F(x, y) = \Re[\bar{z}_1 \varphi_1(z_1) + \chi_1(z_1)]. \quad (8)$$

Based on the relations (5) and (8) we express the stress components by complex potentials $\Phi_1(z_1) = \varphi'_1(z_1)$ and $\Psi_1(z_1) = \psi'_1(z_1) = \chi''(z_1)$ by formulas

$$\begin{cases} \sigma_x = -\gamma^2 \Re[\bar{z}_1 \Phi'_1(z_1) - 2\Phi_1(z_1) + \Psi_1(z_1)], \\ \sigma_y = \Re[\bar{z}_1 \Phi'_1(z_1) + 2\Phi_1(z_1) + \Psi_1(z_1)], \\ \tau_{xy} = \gamma \Im[\bar{z}_1 \Phi'_1(z_1) + \Psi_1(z_1)]. \end{cases} \quad (9)$$

These dependencies can also be denoted in another form (Suo, 1990b; Zhang and Deng, 2008):

$$\begin{cases} \sigma_y + \gamma^{-2} \sigma_x = 4\Re \Phi_1(z_1), \\ \sigma_y - \gamma^{-2} \sigma_x + 2i\gamma^{-1} \tau_{xy} = 2[\bar{z}_1 \Phi'_1(z_1) + \Psi_1(z_1)]. \end{cases} \quad (10)$$

The components u and v of the displacement vector can also be expressed by complex potentials $\varphi_1(z_1)$ and $\psi_1(z_1) = \chi_1(z_1)$ (Prusov and Lunskeya, 1969)

$$2G \left(u + \frac{i}{\gamma} v \right) = \kappa \varphi_1(z_1) - z_1 \overline{\varphi'_1(z_1)} - \overline{\psi_1(z_1)}, \quad (11)$$

where constant κ for plane stress is equal

$$\kappa = \frac{3\sqrt{a_{22}/a_{11}} + a_{12}/a_{22}}{\sqrt{a_{22}/a_{11}} - a_{12}/a_{22}}; \quad \kappa = \frac{3\gamma^2 - \nu_{xy}}{\gamma^2 + \nu_{xy}},$$

for plane strain state:

$$\kappa = 3 - 8G \left(\frac{\nu_{12}}{E_1} + \frac{\nu_{31}\nu_{32}}{E_3} \right).$$

Using the relations (9) one can obtain the following equation (Savruk and Chornenkyi, 2015):

$$2G \frac{d}{dt_1} \left(u + \frac{i}{\gamma} v \right) = \kappa \Phi_1(t_1) - \overline{\Phi_1(t_1)} - \frac{d\bar{t}_1}{dt_1} [t_1 \overline{\Phi'_1(t_1)} + \overline{\Psi_1(t_1)}], \quad t_1 \in L_1, \quad (12)$$

where L_1 – contour on auxiliary plane $z_1 = x + i\gamma y$, which is corresponding to curve L lying on primary complex plane $z = x + iy$.

Let X_n and Y_n be Cartesian components of stress vector acting from the positive direction n on curvilinear contour L . The following relationship bind normal N and tangential T stress components with these components (Muskhelishvili, 2003)

$$X_n + iY_n = -i \frac{dt}{ds} (N + iT) = \frac{d}{ds} \left(\frac{\partial F}{\partial y} - i \frac{\partial F}{\partial x} \right), \quad (13)$$

where s is curvilinear abscissa at the contour L corresponding to the point $t = x + iy$, $t \in L$.

Taking into account Eqs. (8) and (12) we obtain

$$\begin{aligned} \left(\frac{i}{\gamma} X_n - Y_n \right) \frac{ds}{dt_1} &= \Phi_1(t_1) + \overline{\Phi_1(t_1)} \\ &+ \frac{d\bar{t}_1}{dt_1} [t_1 \overline{\Phi'_1(t_1)} + \overline{\Psi_1(t_1)}], \quad t_1 \in L_1. \end{aligned} \quad (14)$$

Eqs. (10) and (12) allow reducing the fundamental problems of the theory of elasticity to the boundary value problems of complex analysis.

If elastic quasi-orthotropic plane is subjected to uniform bidirectional tension of the intensity $\sigma_x^\infty = p$, $\sigma_y^\infty = q$ at the infinity and the uniform shear stress $\tau_{xy}^\infty = \tau$ then the uniform stress state of the plane is defined by the potentials

$$\Phi_1^0(z_1) = \frac{1}{4}(p + q/\gamma^2), \quad \Psi_1^0(z_1) = \frac{1}{2}(p - q/\gamma^2) + i\tau/\gamma. \quad (15)$$

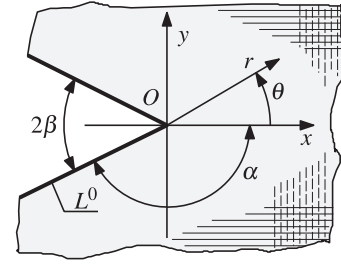


Fig. 1. Infinite elastic quasi-orthotropic wedge with the opening angle 2α .

The normal and tangential components of the stress vector on curvilinear contour L are equal to

$$\begin{aligned} N(t) + iT(t) &= p_0(t) = \frac{1}{2}(p + q) \\ &- \frac{1}{2}(q - p - 2i\tau) \frac{d\bar{t}}{dt}, \quad t = x + iy \in L. \end{aligned} \quad (16)$$

These stresses do not depend on the material parameters and are the same for the case of isotropic plane.

3. Plane eigenvalue problem of the wedge in elastic quasi-orthotropic solid under symmetric loading

Let the elastic wedge (Fig. 1) occupy the area

$$S = \{(r, \theta), r \geq 0, -\alpha \leq \theta \leq \alpha\},$$

where r, θ are polar coordinates with the pole in wedge vertex. The wedge is symmetric with respect to wedge bisector, so we denote the equation describing its contour as $z = x + iy$. On the faces of the wedge there is no normal $\sigma_{\theta\theta}$ or tangential $\tau_{r\theta}$ stresses:

$$\sigma_{\theta\theta} + i\tau_{r\theta} = 0, \quad \theta = \pm\alpha \quad (17)$$

We express the stress function $F_0(x, y)$ for the quasi-orthotropic wedge through two analytical functions $\varphi_1^0(z_1)$ and $\chi_1^0(z_1)$ in the following way:

$$F_0(x, y) = \Re[\bar{z}_1 \varphi_1^0(z_1) + \chi_1^0(z_1)] \quad (18)$$

We take down the stress distribution in the wedge area through stress function expressed in polar coordinates $F_0(r, \theta)$ as (Nowacki, 1970)

$$\begin{aligned} \sigma_{rr} &= \frac{1}{r} \frac{\partial F_0(r, \theta)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F_0(r, \theta)}{\partial \theta^2}, \\ \sigma_{\theta\theta} &= \frac{\partial^2 F_0(r, \theta)}{\partial r^2}, \\ \tau_{r\theta} &= -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial F_0(r, \theta)}{\partial \theta} \right) = \frac{1}{r^2} \frac{\partial F_0(r, \theta)}{\partial \theta} - \frac{1}{r} \frac{\partial^2 F_0(r, \theta)}{\partial r \partial \theta}. \end{aligned} \quad (19)$$

We assume the stress potentials in the form of

$$\varphi_1^0(z_1) = Az_1^{1-\lambda}, \quad \chi_1^0(z_1) = Bz_1^{2-\lambda} \quad (20)$$

where A, B are arbitrary complex constants. The singularity exponent λ is real (Blinowski and Rogaczewski, 2000; Wu and Chang, 1993). The complex variable z_1 was written in the form

$$z_1 = r (\cos^2 \theta + \gamma^2 \sin^2 \theta)^{1/2} \exp[i\beta_1(\theta)], \quad (21)$$

where function $\beta_1(\theta)$ satisfies the equation

$$\tan[\beta_1(\theta)] = \gamma \tan \theta. \quad (22)$$

Thus

$$z_1^{-\lambda} = r^{-\lambda} [R_0(\theta) + iI_0(\theta)], \quad z_1^{2-\lambda} = r^{2-\lambda} [R_1(\theta) + iI_1(\theta)] \quad (23)$$

where

$$\begin{aligned} R_0(\theta) &= (\cos^2 \theta + \gamma^2 \sin^2 \theta)^{-\lambda/2} \cos [\lambda \beta_1(\theta)], \\ I_0(\theta) &= -(\cos^2 \theta + \gamma^2 \sin^2 \theta)^{-\lambda/2} \sin [\lambda \beta_1(\theta)], \\ R_1(\theta) &= (\cos^2 \theta + \gamma^2 \sin^2 \theta)^{1-\lambda/2} \cos [(2-\lambda)\beta_1(\theta)], \\ I_1(\theta) &= (\cos^2 \theta + \gamma^2 \sin^2 \theta)^{1-\lambda/2} \sin [(2-\lambda)\beta_1(\theta)]. \end{aligned} \quad (24)$$

And now, taking into account Eqs. (18) and (19) we find

$$F_0(r, \theta) = r^{2-\lambda} [A_1 \tilde{R}_0(\theta) - A_2 \tilde{I}_0(\theta) + B_1 R_1(\theta) - B_2 I_1(\theta)], \quad (25)$$

where

$$\begin{aligned} A_1 &= \Re A, \quad A_2 = \Im A, \quad B_1 = \Re B, \quad B_2 = \Im B, \\ \tilde{R}_0(\theta) &= R_0(\theta) (\cos^2 \theta + \gamma^2 \sin^2 \theta), \\ \tilde{I}_0(\theta) &= I_0(\theta) (\cos^2 \theta + \gamma^2 \sin^2 \theta). \end{aligned} \quad (26)$$

We obtain stress components from relationships (19) in the form

$$\begin{aligned} \sigma_{rr} &= r^{-\lambda} \{ (2-\lambda) [A_1 \tilde{R}_0(\theta) - A_2 \tilde{I}_0(\theta) + B_1 R_1(\theta) - B_2 I_1(\theta)] \\ &\quad + A_1 \tilde{R}'_0(\theta) - A_2 \tilde{I}'_0(\theta) + B_1 R'_1(\theta) - B_2 I'_1(\theta) \}, \\ \sigma_{\theta\theta} &= r^{-\lambda} (2-\lambda)(1-\lambda) [A_1 \tilde{R}_0(\theta) - A_2 \tilde{I}_0(\theta) + \\ &\quad + B_1 R_1(\theta) - B_2 I_1(\theta)], \\ \tau_{r\theta} &= -r^{-\lambda} (1-\lambda) [A_1 \tilde{R}'_0(\theta) - A_2 \tilde{I}'_0(\theta) + B_1 R'_1(\theta) - B_2 I'_1(\theta)]. \end{aligned} \quad (27)$$

Satisfying the boundary condition (17) and putting $\theta = \alpha$ (see Fig. 1), we obtain the homogeneous system of linear equations, respectively

$$\begin{cases} A_1 R_0(\alpha) (\cos^2 \alpha + \gamma^2 \sin^2 \alpha) + B_1 R_1(\alpha) = 0, \\ A_1 [R'_0(\alpha) (\cos^2 \alpha + \gamma^2 \sin^2 \alpha) + (\gamma^2 - 1) R_0(\alpha) \sin 2\alpha] \\ + B_1 R'_1(\alpha) = 0 \end{cases} \quad (28)$$

for symmetric

$$\begin{cases} A_2 I_0(\alpha) (\cos^2 \alpha + \gamma^2 \sin^2 \alpha) + B_2 I_1(\alpha) = 0, \\ A_2 [I'_0(\alpha) (\cos^2 \alpha + \gamma^2 \sin^2 \alpha) + (\gamma^2 - 1) I_0(\alpha) \sin 2\alpha] \\ + B_2 I'_1(\alpha) = 0 \end{cases} \quad (29)$$

and for anti-symmetric stress distribution with respect to wedge bisector Ox.

Further, we consider the symmetric stress distribution case. The homogeneous system of linear Eq. (28) has a nontrivial solution when its determinant is equal to zero. Hence, the characteristic equation to determine eigenvalues has the following form

$$\sin 2(1-\lambda)\beta_1(\alpha) + (1-\lambda) \sin 2\beta_1(\alpha) = 0, \quad (30)$$

where

$$\beta_1(\alpha) = \pi + \arctan(\gamma \tan \alpha), \quad \pi/2 < \alpha < \pi.$$

In general, the roots of Eq. (30) are complex, but the singular stress field at the wedge vertex raises only if roots of the characteristic equation are in the range $0 < \Re \lambda < 1$. Roots (real or complex) located outside this range can cause singularity in stress derivatives only. Eq. (30) has not any real roots in the interval $0 < \Re \lambda < 1$ for angle α in the range $0 < \alpha < \pi/2$. For wedge angle $\pi/2 < \alpha < \pi$ Eq. (30) has only one real root λ_1 the value of which belongs to the range (0, 1/2). For $\alpha = \pi$ (half-infinite crack) $\lambda_1 = 1/2$. Congruous results were obtained by *Bogy (1972)* and *Wu and Chang (1993)* for orthotropic wedges. Extensive discussion of the solution of the characteristic equation for isotropic wedge were given by *Vasilopoulos (1988)*.

Eq. (30) was received previously by *Savruk and Kazberuk (2014)* in a slightly different form, as the result of limit transition from general orthotropic to quasi-orthotropic material. It is easy to prove the identity of these equations.

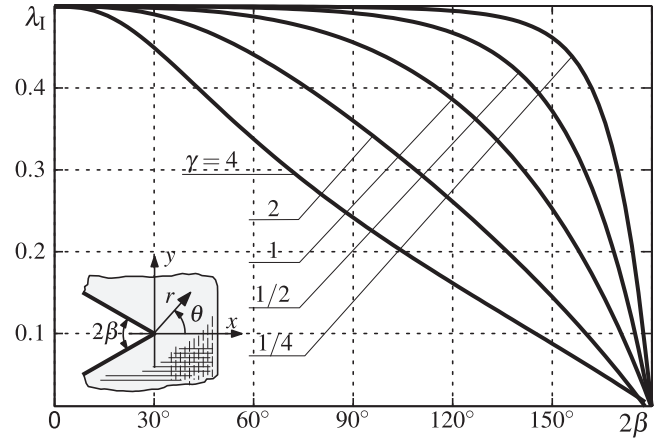


Fig. 2. Stress singularity exponent λ_1 as a function of the notch opening angle 2β for various quasi-orthotropic materials.

It worth noting that characteristic equations for the generally orthotropic wedge with the bisecting line parallel to orthotropy axis were previously constructed by *Blinowski and Rogaczewski (2000)*; *Bogy (1972)*; *Gevorkyan (1968)*; *Selvarathinam and Pageau (1997)*; *Wu and Chang (1993)*; *Wu and Liu (2008)* and *Zappalorto and Carraro (2014)*.

In Fig. 2 we show the numerically obtained dependence of the exponent of the stress singularity λ_1 at the vertex of sharp V-notch in quasi-orthotropic plane as the function of the notch opening angle $2\beta = 2(\pi - \alpha)$ for several γ values. The values of γ parameter $\gamma = 2$ and $\gamma = 4$ correspond to the composite materials fibres of which are arranged parallel to wedge bisector. The values $\gamma = 1/2$ and $\gamma = 1/4$ indicate perpendicular direction of fibres with respect to wedge bisector. The case $\gamma = 1$ – isotropic material – is provided for comparison.

Putting $\beta_1(\alpha) = \alpha$ in Eq. (30), which corresponds to $\gamma = 1$, we obtain well known *Williams (1952)* characteristic equation for wedge eigenproblem in isotropic material. Thereupon, to determine the parameter λ_1 for quasi-orthotropic notch we can use known (see *Savruk and Kazberuk, 2012b*) approximation formula, here expressed as

$$\begin{aligned} \lambda_1 \approx & 1,247 \cos \tilde{\beta} - 1,312 \cos^2 \tilde{\beta} + 0,8532 \cos^3 \tilde{\beta} \\ & - 0,2882 \cos^4 \tilde{\beta}, \quad 0 \leq \tilde{\beta} \leq \pi/2 \end{aligned} \quad (31)$$

where $\tilde{\beta} = \pi - \beta_1(\alpha)$. The maximum absolute error of this estimation is below 0.001 for entire range of parameter $\tilde{\beta}$. Calculated from the (31) relationships between stress singularity exponent λ_1 and notch angle 2β for various values of parameter γ practically coincide with the curves in Fig. 2.

Let us introduce the notch stress intensity factor (NSIF) at the sharp notch vertex in the form (*Carpinteri, 1987*; *Cherepanov, 1979*; *Kazberuk, 2010*; *Savruk and Kazberuk, 2012b*; *Seweryn, 1994*):

$$\tilde{K}_I^V = \lim_{r \rightarrow 0} [(2\pi r)^{\lambda_1} \sigma_{\theta\theta}^0(r, 0)]. \quad (32)$$

It differs from the more common definition of *Gross and Mendelson (1972)* only by multiplier $(2\pi)^{1/2-\lambda_1}$ but it is more convenient (*Carpinteri, 1987*).

Taking into account the relationship (27) we obtain

$$(2 - \lambda_1)(1 - \lambda_1) (2\pi)^{\lambda_1} \{A_1 + B_1\} = \tilde{K}_I^V \quad (33)$$

and joining the first of Eq. (28) we get a system from which we determine constants A_1 and B_2 :

$$\frac{A_1}{R_1(\alpha)} = -\frac{B_1}{\tilde{R}_0(\alpha)} = -\frac{(2\pi)^{-\lambda_1} \tilde{K}_I^V}{(2 - \lambda_1)(1 - \lambda_1) [\tilde{R}_0(\alpha) - R_1(\alpha)]}. \quad (34)$$

From the relationships (20) we get complex potentials in the form

$$\begin{aligned}\Phi_1^0(z_1) &= A_1(1 - \lambda_1)z_1^{-\lambda_1}, \\ \Psi_1^0(z_1) &= B_1(2 - \lambda_1)(1 - \lambda_1)z_1^{-\lambda_1}.\end{aligned}\quad (35)$$

then substituting constants (34) we obtain

$$\begin{aligned}\Phi_1^0(z_1) &= \frac{\tilde{K}_1^V}{(2\pi z_1)^{\lambda_1}} \frac{R_1(\alpha)}{(\lambda_1 - 2)[\tilde{R}_0(\alpha) - R_1(\alpha)]}, \\ \Psi_1^0(z_1) &= \frac{\tilde{K}_1^V}{(2\pi z_1)^{\lambda_1}} \frac{\tilde{R}_0(\alpha)}{\tilde{R}_0(\alpha) - R_1(\alpha)}\end{aligned}\quad (36)$$

Symmetric distribution of the stress field in elastic quasi-orthotropic wedge is defined by relationships (27), thus

$$\begin{aligned}\sigma_{rr}^0 &= \frac{\tilde{K}_1^V}{(2\pi r)^{\lambda_1}} \left\{ \frac{\tilde{R}_0(\alpha)[(2 - \lambda_1)R_1(\theta) + R_1''(\theta)]}{(2 - \lambda_1)(1 - \lambda_1)[\tilde{R}_0(\alpha) - R_1(\alpha)]} \right. \\ &\quad \left. - \frac{R_1''(\alpha)[(2 - \lambda_1)\tilde{R}_0(\theta) + \tilde{R}_0''(\theta)]}{(2 - \lambda_1)(1 - \lambda_1)[\tilde{R}_0(\alpha) - R_1(\alpha)]} \right\}, \\ \sigma_{\theta\theta}^0 &= \frac{\tilde{K}_1^V}{(2\pi r)^{\lambda_1}} \left\{ \frac{\tilde{R}_0(\alpha)R_1(\theta) - R_1(\alpha)\tilde{R}_0(\theta)}{\tilde{R}_0(\alpha) - R_1(\alpha)} \right\}, \\ \tau_{r\theta}^0 &= -\frac{\tilde{K}_1^V}{(2\pi r)^{\lambda_1}} \left\{ \frac{\tilde{R}_0(\alpha)R_1'(\theta) - R_1(\alpha)\tilde{R}_0'(\theta)}{(2 - \lambda_1)[\tilde{R}_0(\alpha) - R_1(\alpha)]} \right\}.\end{aligned}\quad (37)$$

Here the functions $\tilde{R}_0(\theta)$ and $R_1(\theta)$ are given by formulas (24) and (26). Expression (37) defines components of singular stress field in an infinite quasi-orthotropic wedge in the case of symmetrical stress distribution. In the finite bodies with sharp notches these relationships give asymptotic estimation of singular stress distribution in the vicinity of notch vertex.

4. Integral equations of the theory of elasticity for the problem of quasi-orthotropic plane with curvilinear crack

The detailed solution of the problem was already presented in the works of Savruk and Chornenko (2015, 2014b). Here we quote only the basic formulas and equations of the method of singular integral equations necessary for further considerations.

Let us consider at first an auxiliary problem of the theory of elasticity for an quasi-orthotropic plane weakened by a smooth curvilinear crack L . We assume that stresses at the crack edge are continuous while displacements are discontinuous:

$$[(i/\gamma)X_n - Y_n]^+ - [(i/\gamma)X_n - Y_n]^- = 0, \quad t \in L, \quad (38)$$

$$[u + (i/\gamma)v]^+ - [u + (i/\gamma)v]^- = \frac{4i\gamma^2}{E_x} g(t), \quad t \in L, \quad (39)$$

and stresses and rotation in infinity are considered null. Upper indexes “+” or “−” indicate boundary values of corresponding variables when $z \rightarrow t \in L$ from the left (+) or from the right (−) in relation to selected positive direction of L , respectively.

After differentiation Eq. (39) takes the form

$$\frac{d}{dt_1} \left[\left(u + \frac{i}{\gamma} v \right)^+ - \left(u + \frac{i}{\gamma} v \right)^- \right] = \frac{4i\gamma^2}{E_x} g'_1(t_1) \quad (40)$$

where $g_1(t_1) = g(t)$, $t_1 = x + iy \in L_1$.

Using the relationships (12), (14), (38) and (40), the boundary value problem can be denoted as the system

$$\begin{cases} \Phi_1^+(t_1) - \Phi_1^-(t_1) = ig'_1(t_1), \\ [\bar{t}_1 \Phi_1'(t_1) + \Psi_1(t_1)]^+ - [\bar{t}_1 \Phi_1'(t_1) + \Psi_1(t_1)]^- \\ = i[\overline{g'_1(t_1)} - g'_1(t_1)] \frac{d\bar{t}_1}{dt_1}, \quad t_1 \in L_1, \end{cases}$$

the solution of which is known (Savruk, 1981):

$$\begin{aligned}\Phi_1(z_1) &= \frac{1}{2\pi} \int_{L_1} \frac{g'_1(t_1) dt_1}{t_1 - z_1}, \\ \Psi_1(z_1) &= \frac{1}{2\pi} \int_{L_1} \left[\frac{\overline{g'_1(t_1)} d\bar{t}_1}{t_1 - z_1} - \frac{\bar{t}_1 g'_1(t_1) dt_1}{(t_1 - z_1)^2} \right]\end{aligned}\quad (41)$$

Expression (41) can be considered as the integral representation of the complex stress potentials $\Phi_1(z_1)$ and $\Psi_1(z_1)$ expressed by the derivative of the step displacement function over the curvilinear contour L , while stresses are continuous.

Based on the complex representation of potentials (41), various boundary value problems for quasi-orthotropic bodies with crack and openings can be solved (Savruk, 1981).

Let the self-balanced stresses on the edges of the crack along the contour L be set:

$$N^+ + iT^+ = N^- + iT^- = p(t), \quad t \in L, \quad (42)$$

while stresses at infinity disappear. The condition (42) can also be written as

$$\begin{aligned}[(i/\gamma)X_n^\pm - Y_n^\pm] \frac{ds}{dt_1} &= \tilde{p}(t) = \tilde{p}_1(t_1) \\ &= \frac{1}{2\gamma} \left[(1 + \gamma)p(t) - (1 - \gamma)\overline{p(t)} \frac{d\bar{t}}{dt} \right] \frac{dt}{dt_1}\end{aligned}\quad (43)$$

Satisfying the boundary condition (43) by potentials (41) to determine the unknown function $g'_1(t_1)$, we obtain singular integral equation

$$\frac{1}{\pi} \int_{L_1} [K_1(\tau_1, t_1) g'_1(\tau_1) d\tau_1 + L_1(\tau_1, t_1) \overline{g'_1(\tau_1)} d\bar{\tau}_1] = \tilde{p}_1(t_1), \quad (44)$$

where

$$\begin{aligned}K_1(\tau_1, t_1) &= \frac{1}{2} \left[\frac{1}{\tau_1 - t_1} + \frac{1}{\bar{\tau}_1 - \bar{t}_1} \frac{d\bar{t}_1}{dt_1} \right], \\ L_1(\tau_1, t_1) &= \frac{1}{2} \left[\frac{1}{\bar{\tau}_1 - \bar{t}_1} - \frac{\tau_1 - t_1}{(\bar{\tau}_1 - \bar{t}_1)^2} \frac{d\bar{t}_1}{dt_1} \right].\end{aligned}\quad (45)$$

Note that the integral Eq. (44) is consistent with the known one for degenerate anisotropic body (Savruk and Kazberuk, 2014) that was obtained by limit transition from the general case of anisotropic plane with curved cracks.

The solution of Eq. (44) should satisfy additional condition

$$\int_{L_1} g'_1(t_1) dt_1 = 0, \quad (46)$$

which ensures unambiguity of displacements during tracing the contour L .

The complex stress potentials (41) and singular integral Eq. (44) are valid also for systems of curvilinear cracks in quasi-orthotropic plane. Then, the symbol L denotes the set of crack contours and the additional condition (46) must be fulfilled for each contour component separately.

5. Rounded infinite V-notch

Let us consider the elastic quasi-orthotropic plane weakened by infinite rounded V-shaped notch with the opening angle equal to

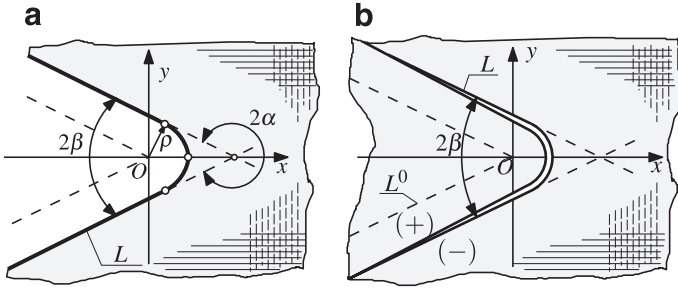


Fig. 3. a) Infinite rounded V-notch with edge contour L and b) infinite crack in quasi-orthotropic plane along contour L .

2β (Fig. 3a). We assume that the edge L of the elastic plane is free of load and the asymptotic stress field at infinity (37) is described by the complex potentials (36). This problem can also be interpreted in a slightly different way. In the elastic wedge with boundary contour L_0 and stress and deformation state described by the complex potentials (36), there is the cut (crack) along the contour L (Fig. 3b). We need to find the disturbance of the stress field in the elastic plane (or wedge) induced by the presence of rounded notch with boundary contour L . Below, we consider a symmetrical problem when the stress state is symmetric with respect to the Ox axis.

We solve the problem using a superposition method. We are looking for complex potentials in the form of

$$\Phi_1(z_1) = \Phi_1^0(z_1) + \frac{1}{2\pi} \int_{L_1} \frac{g'_1(t_1) dt_1}{t_1 - z_1} \quad (47)$$

$$\Psi_1(z_1) = \Psi_1^0(z_1) + \frac{1}{2\pi} \int_{L_1} \left[\frac{g'_1(t_1) d\bar{t}_1}{t_1 - z_1} - \frac{\bar{t}_1 g'_1(t_1) dt_1}{(t_1 - z_1)^2} \right], \quad (48)$$

where functions $\Phi_1^0(z_1)$ and $\Psi_1^0(z_1)$ (36) describe the basic stress state in quasi-orthotropic wedges.

For determination of unknown function $g'_1(t_1)$ we use singular integral Eq. (44), the right side of which is

$$\tilde{P}_1(t_1) = - \left[\Phi_1^0(t_1) + \overline{\Phi_1^0(t_1)} + \frac{dt_1}{dt_1} (t_1 \overline{\Phi_1^0(t_1)} + \overline{\Psi_1^0(t_1)}) \right], t_1 \in L_1. \quad (49)$$

We define the contour L of the rounded V-shaped notch by means of the following parametric equation

$$t = \omega(\xi) = \rho t = \omega(\xi) \begin{cases} \sin \beta + (\zeta + \zeta_B) \cos \beta + \\ -i[\cos \beta - (\zeta + \zeta_B) \sin \beta], & -\infty < \zeta < -\zeta_B, \\ \cos \zeta + i \sin \zeta, & -\zeta_B \leq \zeta \leq \zeta_B, \\ \sin \beta - (\zeta - \zeta_B) \cos \beta + \\ +i[\cos \beta + (\zeta - \zeta_B) \sin \beta], & \zeta_B < \zeta < \infty, \end{cases} \quad (50)$$

where $\zeta = \xi/(1 - \xi^2)^c$, $\zeta_B = \pi/2 - \beta$ is an angular coordinate of the circular arc end point, c is an arbitrary positive number (here $c = 3/2$), chosen experimentally to obtain the best convergence of the solution.

We also consider contour L_H as an edge of hyperbolic notch (Benthem, 1987) in quasi-orthotropic plane with the same radius of curvature at the notch vertex. Parametric equation of the contour L_H is

$$\omega(\xi) = \rho \left(\frac{e^{i\xi\alpha} \cos \alpha}{\cos \alpha - \cos \xi\alpha} + \frac{1}{2} \cot^2 \frac{\alpha}{2} \right), \alpha = \pi - \beta, -1 < \xi < 1. \quad (51)$$

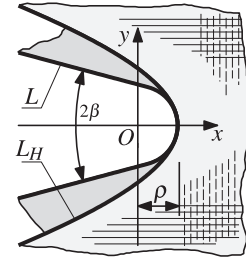


Fig. 4. Comparison of the shape of rounded V-notch L and hyperbolic notch L_H of the same radius of curvature at the notch vertex.

The comparison of the notch contours in the vicinity of notch apex is shown in Fig. 4.

The parametric equations of the contours related to the auxiliary plane z_1 have the form

$$t_1 = \omega_1(\xi) = \frac{1}{2} [(1 + \gamma)\omega(\xi) + (1 - \gamma)\overline{\omega(\xi)}], \quad -1 < \xi < 1. \quad (52)$$

Substituting

$$\tau = \omega(\xi), \quad t = \omega(\eta), \quad \tau_1 = \omega_1(\xi), \quad t_1 = \omega_1(\eta) \quad (53)$$

and denoting

$$g'_1(\omega_1(\xi)) \omega'_1(\xi) = g'_1(\xi)$$

we express the integral Eq. (44) and condition (46) in the canonical form

$$\begin{cases} \frac{1}{\pi} \int_{-1}^{+1} [\tilde{K}_1(\xi, \eta) g'_1(\xi) + \tilde{L}_1(\xi, \eta) \overline{g'_1(\xi)}] d\xi = \tilde{P}_1(\eta), \\ -1 < \eta < 1, \\ \int_{-1}^{+1} g'_1(\xi) d\xi = 0, \end{cases} \quad (54)$$

where

$$\tilde{P}_1(\eta) = \tilde{P}_1(\omega_1(\eta)),$$

$$K_1(\xi, \eta) = K_1(\omega_1(\xi), \omega_1(\eta)),$$

$$L_1(\xi, \eta) = L_1(\omega_1(\xi), \omega_1(\eta)).$$

We look for the unknown function $g'_1(\xi)$ in the class of functions with integrable singularity on the endpoints of the integration interval

$$g'_1(\xi) = \frac{u_1(\xi)}{\sqrt{1 - \xi^2}}, \quad (55)$$

where $u_1(\xi)$ – continuous function over $-1 \leq \xi \leq 1$.

We find the numerical solution of the integral Eq. (54) using the method of mechanical quadratures (see e.g. Erdogan et al., 1973; Linkov, 2002; Savruk, 1981; Savruk et al., 1989) and Gauss–Chebyshev quadrature. As the result we obtain system of $2n$ linear equations with $2n$ unknowns $u_1(\xi_k)$ ($k = 1, \dots, 2n$):

$$\begin{cases} \frac{1}{2n} \sum_{k=1}^{2n} [K_1(\xi_k, \eta_m) u_1(\xi_k) + L_1(\xi_k, \eta_m) \overline{u_1(\xi_k)}] = \tilde{P}_1(\eta_m), \\ \sum_{k=1}^{2n} u_1(\xi_k) = 0, \end{cases} \quad (56)$$

where coordinates of quadrature nodes and collocation points are given by the formulas

$$\begin{aligned} \xi_k &= \cos \frac{\pi(2k-1)}{4n}, \quad k = 1, \dots, 2n, \\ \eta_m &= \cos \frac{\pi m}{2n}, \quad m = 1, \dots, 2n-1. \end{aligned} \quad (57)$$

The symmetry of the stress field (with respect to the Ox axis) implies condition

$$u_1(-\xi) = \overline{u_1(\xi)}, \quad (58)$$

which gives us the opportunity to reduce the order of linear system (56) by two. As the result, we obtain

$$\begin{cases} \frac{1}{2n} \sum_{k=1}^n [M(\xi_k, \eta_m) u_1(\xi_k) + N(\xi_k, \eta_m) \overline{u_1(\xi_k)}] = \tilde{P}_1(\eta_m), \\ \sum_{k=1}^n [u_1(\xi_k) + \overline{u_1(\xi_k)}] = 0, \end{cases} \quad (59)$$

where

$$\begin{aligned} M(\xi_k, \eta_m) &= K_1(\xi_k, \eta_m) + L_1(-\xi_k, \eta_m), \\ N(\xi_k, \eta_m) &= L_1(\xi_k, \eta_m) + K_1(-\xi_k, \eta_m). \end{aligned} \quad (60)$$

The stress components on crack contour L can be calculated according to the formulas (9). We calculate the Cauchy principal values of the potentials (47) analogously to isotropic case (Savruk, 1981). Since boundary contour is free of external load ($\sigma_n = \tau_{ns} = 0$) and the sum of normal stresses is invariant i.e. $\sigma_n + \sigma_s = \sigma_x + \sigma_y$, then the normal stress along the edge of the notch is equal

$$\sigma_s(\eta) = \sigma_s^-(\eta) = \sigma_x^-(\eta) + \sigma_y^-(\eta) = (2\pi\rho)^{-\lambda_1} \tilde{K}_1^V R_1(\eta), \quad (61)$$

where $R_1(\eta)$ – dimensionless normal stress at point η .

At the rounded vertex of the V-notch dimensionless normal stress $R_1(0)$ is equal R_1 . It is so called the stress rounding factor (Benthem, 1987; Savruk and Kazberuk, 2006). Its values depend (for isotropic material) on the notch opening angle and the notch shape in a neighborhood of the vertex (Savruk and Kazberuk, 2006). The result is the relationship

$$\sigma_s(0) = (2\pi\rho)^{-\lambda_1} \tilde{K}_1^V R_1 \quad (62)$$

that is asymptotic for finite bodies with V-notches rounded with small radius of curvature ρ at the notch vertex. As it was earlier demonstrated (Savruk and Kazberuk, 2007, 2010, 2012b), the relationship (62) can be used to estimate the maximum stress at the vertex of V-notch rounded with small radius of curvature (when the NSIF is known), or we can calculate the notch stress intensity factor at the vertex of sharp notch:

$$\tilde{K}_1^V = \frac{(2\pi)^{\lambda_1}}{R_1} \lim_{\rho \rightarrow 0} \sigma_s(0) \rho^{\lambda_1}, \quad (63)$$

performing the calculations for stress concentration at the rounded vertex of V-notch of the same opening angle.

6. Numerical results

6.1. Stress concentration at the vertex of rounded infinite V-notch in quasi-orthotropic plane

The numerical calculations were performed for quasi-orthotropic plate (plane stress) with rounded V-shaped or hyperbolic notch. The discrete values of the material parameter γ were selected from the interval $1/4 \leq \gamma \leq 4$. The distributions of the dimensionless traction stress $R_1(\theta) = R_1(\eta)$, ($\theta = \text{Arg}\omega(\eta) \in [-\pi, \pi]$) alongside rounded V-notch contour in quasi-orthotropic plane are depicted in Fig. 5 for notch opening angle $2\beta = \pi/3$. It is worth noting that for values $\gamma \geq 1$ ($E_x = E_1$ – fibres parallel to the notch bisector) function $R_1(\theta)$ may have either a local maximum or a minimum at the notch vertex ($\theta = 0$) depending on γ value. The maximum value of $R_1(\theta)$ occurs at some distance from the vertex. For $\gamma < 1$ ($E_x = E_2$ – fibres perpendicular to the notch bisector) function $R_1(\theta)$ has maximum at the notch vertex.

The presence of local maximum or minimum of stress function $R_1(\theta)$ at the notch vertex depends on value of material parameter γ , notch opening angle β and notch shape. In Fig. 6 we show distribution of $R_1(\theta)$ for several values of angle 2β at constant orthotropy parameter value $\gamma = 4$. For $2\beta = 1^\circ$ the notch shape is almost parabolic and calculated $R_1(\theta)$ values agree well

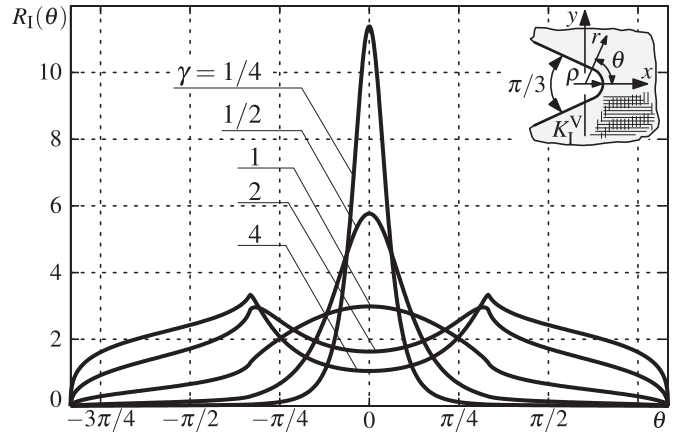


Fig. 5. Distribution of the dimensionless traction stress $R_1(\theta)$ alongside the contour of rounded infinite V-notch with the opening angle $2\beta = 60^\circ$ for various values of quasi-orthotropy parameter γ .

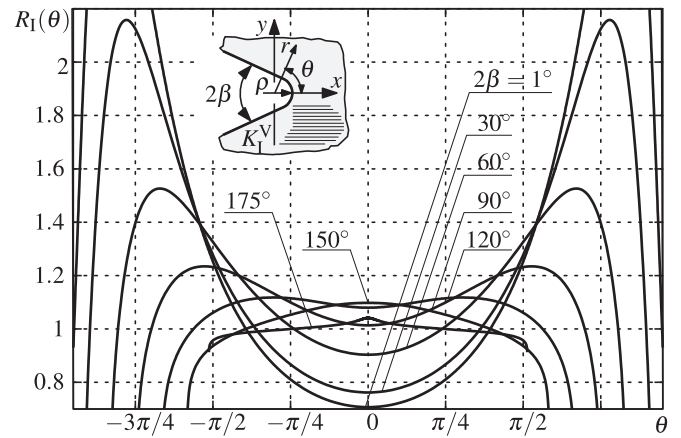


Fig. 6. Distribution of the dimensionless traction stress $R_1(\theta)$ alongside the contour of hyperbolic notch at parameter $\gamma = 4$ for various values of the opening angle 2β .

with purely analytical solution (Savruk et al., 2015), which was obtained as limit case of elliptic hole problem (see Lekhnitskii, 1981). The curve plotted for $2\beta = 175^\circ$ tends to a constant value: $R_1(\theta) \rightarrow 1$, which corresponds to stress state at the edge of half-plane under unidirectional tension.

The relationship between stress rounding factor $R_1 = R_1(0)$ and F notch opening angle 2β for various values of the material parameter γ and two considered contours of the notch is shown in Fig. 7. The value of the stress rounding factor R_1 is significantly affected by the direction on the notch bisector relative to principal material axes of orthotropy (fibres parallel – $E_x = E_1$, $\gamma > 1$ or perpendicular – $E_x = E_2$, $\gamma < 1$). Curve $\gamma = 1$ corresponds to isotropic material. As it can be seen, the relative difference between R_1 values calculated for rounded V-notch (50) and hyperbolic notch (51) is rather small. For the angle $2\beta = 0$ and isotropic material ($\gamma = 1$) this difference is about 6% (Savruk and Kazberuk, 2006) and decreases with the decrease of parameter γ . For $\gamma > 1$ the relative difference increases, reaching 24% for $\gamma = 4$ and $2\beta = 0$.

The desired accuracy of the R_1 calculations was achieved using simple self-adaptive scheme. The result was constructed as a sequence of solutions obtained by consecutively doubling the number of quadrature nodes (57) n starting from $n = 64$. The process was stopped when the relative difference between R_1 value calculated at previous step (n nodes) and the current result (for $2n$ nodes) was less than 0.1%. For small values of γ and 2β the maximum value of n was 16384.

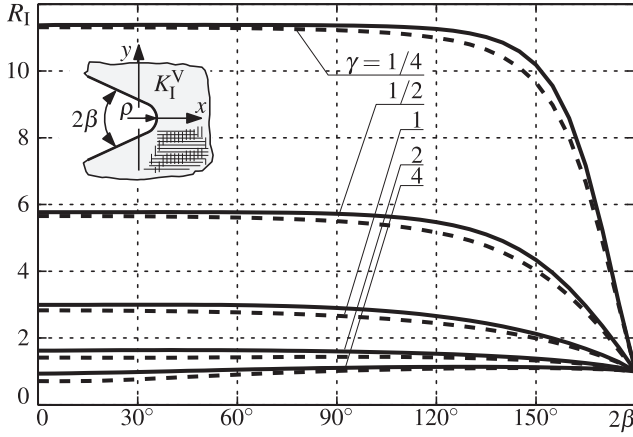


Fig. 7. Stress rounding factor R_I as the function of notch opening angle 2β for rounded V-notch (solid lines) and hyperbolic notch (dashed lines).

Table 1

Stress intensity exponents λ_1 and stress rounding factors R_I for rounded V-shaped notch in quasi-orthotropic plane.

2β	$\gamma = 1/4$		$\gamma = 1/2$		$\gamma = 2$		$\gamma = 4$	
	λ_1	R_I	λ_1	R_I	λ_1	R_I	λ_1	R_I
0°	0.5000	11.371	0.5000	5.770	0.5000	1.620	0.5000	0.932
10°	0.5000	11.383	0.5000	5.770	0.4996	1.625	0.4969	0.944
20°	0.5000	11.383	0.4999	5.771	0.4968	1.632	0.4798	0.962
30°	0.5000	11.384	0.4998	5.772	0.4899	1.637	0.4488	0.982
40°	0.4999	11.384	0.4995	5.773	0.4782	1.639	0.4110	1.004
50°	0.4999	11.384	0.4990	5.772	0.4616	1.637	0.3720	1.027
60°	0.4998	11.383	0.4982	5.768	0.4408	1.632	0.3345	1.050
70°	0.4996	11.381	0.4969	5.761	0.4165	1.624	0.2993	1.071
80°	0.4993	11.377	0.4948	5.747	0.3894	1.613	0.2666	1.089
90°	0.4988	11.368	0.4916	5.722	0.3600	1.599	0.2362	1.105
100°	0.4980	11.352	0.4867	5.678	0.3287	1.581	0.2075	1.119
110°	0.4967	11.322	0.4792	5.602	0.2958	1.558	0.1803	1.129
120°	0.4943	11.261	0.4674	5.474	0.2611	1.529	0.1541	1.135
130°	0.4899	11.133	0.4487	5.260	0.2245	1.492	0.1286	1.137
140°	0.4811	10.851	0.4191	4.905	0.1859	1.444	0.1035	1.134
150°	0.4616	10.186	0.3719	4.340	0.1448	1.383	0.0785	1.124
160°	0.4147	8.590	0.2971	3.496	0.1006	1.301	0.0531	1.105
170°	0.2957	5.239	0.1802	2.360	0.0527	1.189	0.0271	1.072
180°	0.0000	1.000	0.0000	1.000	0.0000	1.000	0.0000	1.000

The smallest 2β angle for which the results of R_I were calculated using procedure described in Section 5 was $2\beta = 1^\circ$. For $2\beta = 0$ (semi-infinite U-shaped slot) the values of R_I were obtained by extrapolation. These results were used to establish simple formula for estimation of the stress rounding factor for U-shaped notch in quasi-orthotropic body:

$$R_I(\beta = 0) \approx 0.245 + \frac{2.741}{\gamma} + \left(\frac{0.102}{\gamma}\right)^2$$

with relative error less than 0.2% in the range $1/4 \leq \gamma \leq 4$.

The contour of the hyperbolic notch for $2\beta \rightarrow 0$ approaches the parabola. The values of R_I calculated for $2\beta = 1.0^\circ$ agree well (relative difference is less than 0.1%) with the results obtained analytically for the problem of elliptical hole in orthotropic plane subjected to uniform tension (Chiang, 1994; Lekhnitskii, 1981):

$$R_H(\beta = 0) = 2\sqrt{2}/\gamma.$$

To calculate notch stress intensity factors K_I^V at the sharp vertex of a notch using the relationship (63) one needs to know not only stress rounding factor R_I but the stress singularity exponent λ_1 as well. Both of these parameters are shown in Table 1 for selected values of orthotropy parameter γ .

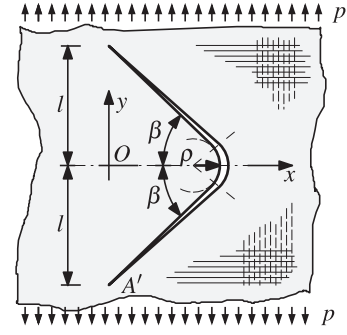


Fig. 8. Chevron crack with rounded corner in quasi-orthotropic plane under unidirectional tension.

6.2. Stress intensity factor at the sharp corner of chevron crack

Let us consider an elastic quasi-orthotropic plane with a smooth curvilinear crack (Fig. 8). The contour L of the crack consists of two straight edges connected with circular arc. The angle between crack arms is equal 2β . The Ox axis is the axis of symmetry of the crack and it is parallel to the principal material axes. The projection of the crack on the Oy axis is equal 2ℓ , where ℓ is the main length unit. The radius of the arc is ρ . We define the relative radius of curvature of rounded edge at vertex as $\varepsilon = \rho/\ell$. The center of the circle, the part of which is vertex arc, is located at the point $z = \ell(\cos\beta - \varepsilon)/\sin\beta$. Therefore, when ε approaches zero we obtain sharp chevron (or kinked) crack. For this limit case ($\rho \rightarrow 0$), the length of the crack arm is equal $a = \ell/\sin\beta$.

The parametric equation $t = \omega(\xi)$, $-1 \leq \xi \leq 1$, that describes contour L has the following form:

$$\omega(\xi) = \ell \begin{cases} (1 - \varepsilon \cos \beta) / \tan \beta + c(\tau + \tau_B) \cos \beta \\ -i[\varepsilon \cos \beta - c(\tau + \tau_B) \sin \beta], & -1 \leq \tau < -\tau_B, \\ (\cos \beta - \varepsilon) / \sin \beta + \varepsilon \cos(\tau c / \varepsilon) \\ + i \varepsilon \sin(\tau c / \varepsilon), & -\tau_B \leq \tau \leq \tau_B, \\ (1 - \varepsilon \cos \beta) / \tan \beta - c(\tau - \tau_B) \cos \beta \\ + i[\varepsilon \cos \beta + c(\tau - \tau_B) \sin \beta], & \tau_B < \tau < 1, \end{cases} \quad (64)$$

where

$$\tau = \mathfrak{S}(\xi) = \varepsilon \sinh(\xi \operatorname{arsinh}(\varepsilon^{-1})),$$

and $\mathfrak{S}(\xi)$ is the nonlinear function of coordinates transformation (Johnston et al., 2007). Parameters τ_B and c are equal:

$$\tau_B = \varepsilon(\pi/2 - \beta)/c,$$

$$c = (1 - \varepsilon \cos \beta) / \sin \beta + \varepsilon(\pi/2 - \beta) \quad \varepsilon = \rho/\ell.$$

The edges of the crack are free of load. The plane at the infinity is subjected to uniform unidirectional tension, the intensity of which is $\sigma_y^\infty = p$. Thus, the potentials (15) describing the uniform stress field are equal

$$\Phi_1^0(z_1) = \frac{1}{4}p, \quad \Psi_1^0(z_1) = \frac{1}{2}p.$$

Thus, to obtain the right-hand side of Eq. (44) we substitute the expression (16) into (43) in the following form:

$$p_0(t) = -\frac{1}{2}p \left(1 + \frac{d\bar{t}}{dt}\right), \quad t \in L.$$

The same crack in isotropic plane was analyzed by Theocaris and Ioakimidis (1980). They calculated the crack stress intensity factors at the end tips of the crack and the notch stress intensity factor at the sharp corner. Here we concentrate on the influence of

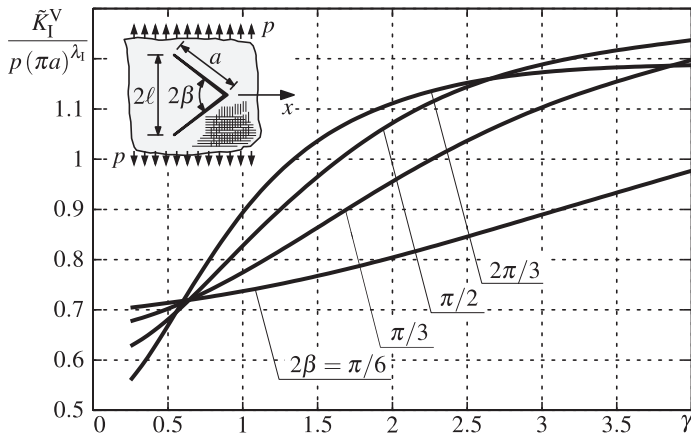


Fig. 9. Notch stress intensity factor at the corner point of chevron crack as a function of orthotropy parameter γ .

orthotropy parameter γ on NSIF at the sharp corner. The stresses at the vertex were calculated for several values of the crack angle $2\beta = \{30^\circ, 60^\circ, 90^\circ, 120^\circ\}$. The rounded notch relative radius range was $0.0001 < \varepsilon < 0.5$. The material parameter γ was in the range $1/4 \leq \gamma \leq 4$. The calculated values of the notch stress intensity factor in dimensionless form are presented in Fig. 9. For isotropic material ($\gamma = 1$) the values of NSIF agree well with the results already published by Theocaris and Ioakimidis (1980). It is evident that for $2\beta \gg 0$ the notch stress intensity factor $\tilde{K}_I^V$ strongly depends on the orientation of the crack relative to fibres of quasi-orthotropic composite ($\gamma < 1$ or $\gamma > 1$) and the magnitude of material parameter γ . For $2\beta \rightarrow 0$ NSIF values approach the constant value $\tilde{K}_I^V \rightarrow K_I = (\sqrt{2}/2)p\sqrt{\pi a}$, i.e. to value of SIF for rectilinear crack, which does not depend on material properties.

7. Conclusions

The boundary value problem of semi-infinite rounded V-notch in quasi-orthotropic plane subjected to symmetrical deformation (mode I) was solved. The relationships between the stress intensity factor for sharp V-notch and stress concentration factor at the vertex of rounded V-shaped or hyperbolic notch (with the same opening angle) were established. It was shown that the influence of the notch shape on the stress concentration factor at the notch vertex depends on the value of orthotropy parameter γ . For $\gamma \ll 1$ (unidirectional composite with fibres perpendicular to the notch bisector) and small notch angle 2β the relative difference between rounded V-notch and hyperbolic notch is negligible. Otherwise, for $\gamma > 1$ (fibres are arranged along the notch bisector) the relative difference increases with increasing of parameter γ . It is also worth noting that with the increase of orthotropy parameter the location of the maximum (from the notch vertex towards the lateral edge of the notch) also changes.

The obtained relationship between the stress intensity factor for sharp V-notch and stress concentration factor at the vertex of rounded V-notch has asymptotic character for finite bodies with notches and it can be used to calculate NSIF based on stress concentration results obtained for notch rounded with small radius of curvature. The effectiveness of this approach was shown on the example of chevron crack in unidirectional stretched quasi-orthotropic plane. The calculated values of the notch stress intensity factor at the crack corner strongly depend on the value of the orthotropy parameter. What is also important, the single orthotropy parameter γ comprising ratio of the two Young's moduli is a convenient analytical factor in the investigation of the evolution of the stress field in orthotropic medium.

Acknowledgments

This work was supported by National Science Center (Poland) under the project 2011/03/B/ST8/06456.

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