

PERIODIC SYSTEM OF CLOSELY LOCATED HOLES IN AN ELASTIC PLANE UNDER THE CONDITIONS OF ANTIPLANE DEFORMATION

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We construct a singular integral equation of an antiplane periodic problem of the theory of elasticity for an isotropic plane weakened by smooth curvilinear holes. A numerical solution of the problem is obtained in the case of closely located holes with rounded V-shaped vertices made in the elastic plane under the conditions of uniform shear at infinity. On this basis, we determine the stress concentration factors at the rounded vertices of a bilateral V-shaped semiinfinite notch. By using the well-known relation between the stress intensity and stress concentration factors for sharp and rounded V-shaped notches, we perform the limit transition to a bilateral sharp V-shaped notch. The dependence of the stress intensity factor at the sharp vertices of a bilateral V-notch on its apex angle is determined.

Keywords: antiplane problem of the elasticity theory, periodic system of curvilinear holes, bilateral V-shaped notch, stress intensity factors, method of singular integral equations.

In fracture mechanics, a significant attention is now given to the investigation of the processes of deformation and fracture of solids with sharp and rounded V-shaped notches [1–5]. The two-dimensional problems of the elasticity theory for bodies with V-shaped notches can be solved within the framework of a single approach developed in [1, 2, 5–10] for the modes I and II of deformation, and in [11, 12] for the mode III of deformation. According to this approach, the stress intensity factors at the tip of a sharp notch are determined on the basis of the data on stress concentration of the tips of the corresponding rounded V-shaped notches whose radii of curvature are not too small. These data can be obtained by using various methods. The relationships between the stress intensity factors at the sharp tip of a semiinfinite V-shaped notch and the maximum stresses on the contour of the corresponding rounded V-shaped notch have been constructed. For bounded bodies, these relationships have the asymptotic character (as the radius of curvature at the tip of the notch tends to zero). This enables one to use these relations for the determination of the stress intensity factors at the tips of sharp notches according to the data on the concentration of stresses near rounded notches in the elastic bodies of any shape.

In what follows, by the method of singular integral equations, we obtain a numerical solution of the antiplane problem for a periodic system of closely located curvilinear holes with rounded V-shaped tips. On this basis, we find the stress concentration factors at the tips of a two-sided V-shaped notch. By using the well-known relations between the stress intensity and stress concentration factors for sharp and rounded V-shaped notches [11, 12], we establish the dependence of the stress intensity factors (SIF) at the tip of a two-sided sharp V-shaped notch on its apex angle under the conditions of antiplane deformation with the help of the limit transition as the radius of curvature of the notch tends to zero.

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Basic Relations of the Antiplane Theory of Elasticity [2, 5, 13]

In a cylindrical body, we consider a stressed state induced by a load directed along the generatrices of the cylinder and constant along these generatrices. If the axis of deformation is directed along the Oz -axis of a Cartesian coordinate system (x, y, z) , then, by virtue of Hooke's law, the nonzero components of the stress tensor τ_{xz} and τ_{yz} are functions of x and y and can be expressed via the displacements $w(x, y)$ as follows:

$$\tau_{xz} = G \frac{\partial w}{\partial x}, \quad \tau_{yz} = G \frac{\partial w}{\partial y}, \quad (1)$$

where G is the shear modulus.

The function $Gw(x, y)$ satisfies the Laplace equation and can be represented in the form of the real part of an analytic function $f(z)$ of complex variable $z = x + iy$:

$$Gw(x, y) = \operatorname{Re} f(z). \quad (2)$$

The function $f(z)$ is expressed via the components of stresses as follows:

$$\tau_{xz} - i\tau_{yz} = f'(z) = F(z). \quad (3)$$

The solution of the antiplane problem of the elasticity theory can be reduced to the determination of the analytic function $f(z)$ or $F(z)$ in the region S occupied by the elastic body with regard for the given boundary values of these functions on a contour L (boundary of the body). For the first basic problem, when external loads are given on the contour, the boundary condition takes the form

$$\tau_{nz} = G \frac{\partial w}{\partial n} = \operatorname{Im} \left[F(t) \frac{dt}{ds} \right] = \tau(t), \quad t \in L, \quad (4)$$

where $\tau(t)$ is a given stress τ_{nz} on the boundary of the body L with the outer normal n and s is the arc abscissa of the point t .

Assume that the infinite body has a curvilinear tunnel cut L whose faces are loaded by self-balanced forces

$$\tau_{nz}^+ = \tau_{nz}^- = \tau(t), \quad t \in L, \quad (5)$$

and that the stresses at infinity are absent. In relation (5), the superscripts “+” and “−” denote the limiting values of the functions on the contour L , as $z \rightarrow t$ from the left (+) and from the right (−) relative to the chosen positive direction, respectively.

For the cut, we take the complex potential of stresses in the form

$$F(z) = \frac{1}{\pi i} \int_L \frac{\gamma'(t) dt}{t - z}, \quad (6)$$

where $\gamma'(t)$ is the derivative of the jump of displacements on the contour L :

$$G(w^+ - w^-) = 2\gamma(t). \quad (7)$$

Satisfying the boundary condition (5) with the help of the integral representation (6), we arrive at the following singular integral equation:

$$\frac{1}{\pi} \int_L K(t, t') \gamma'(t) dt = \tau(t'), \quad t' \in L, \quad (8)$$

where

$$K(t, t') = \operatorname{Re} \left[\frac{1}{t - t'} \frac{dt'}{ds'} \right]. \quad (9)$$

In the class of functions with integrable singularity at the ends of the contour of integration, the integral equation (8) possesses a unique solution under the following additional condition:

$$\int_L \gamma'(t) dt = 0. \quad (10)$$

This condition guarantees the uniqueness of displacements in traversing the crack contour.

If the elastic plane is weakened by a periodic system of curvilinear cracks located along the Ox -axis with period d and the same load (5) acts upon the crack faces, then the integral representation of the complex potential has the form

$$F(z) = \frac{1}{id} \int_L \gamma'(t) \cot \frac{\pi}{d} (t - z) dt, \quad (11)$$

where L is the crack contour in the main band of periods. In this case, we arrive at the integral equation (9) with the kernel

$$K(t, t') = \frac{\pi}{d} \operatorname{Re} \left[\cot \frac{\pi}{d} (t - t') \frac{dt'}{ds'} \right]. \quad (12)$$

Note that, for the uniqueness of the solution of the singular integral equation (8), it is necessary to impose the additional condition (10).

Periodic System of Curvilinear Holes

Consider an elastic body with periodic row of identical curvilinear holes uniformly arranged along the Ox axis with period d . By L we denote the boundary contour of a hole within the main band of periods. We say that the direction of traversing the contour is positive if the elastic region S remains to the left (Fig. 1).

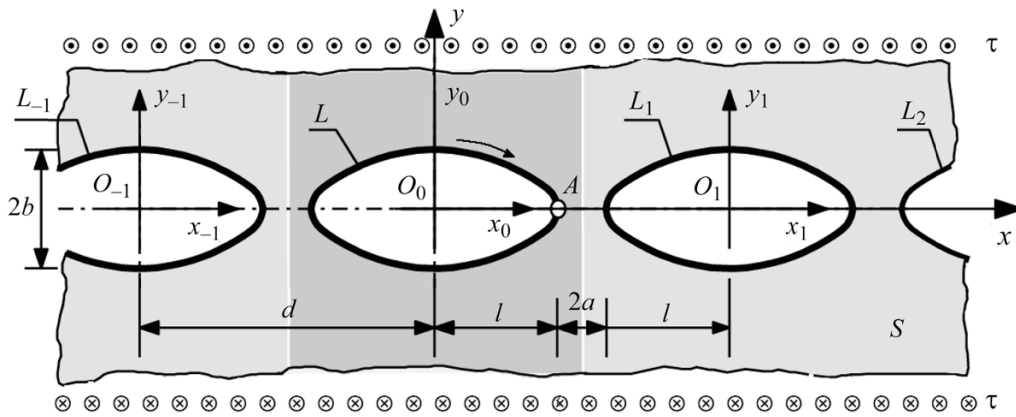


Fig. 1. Periodic system of curvilinear holes in the elastic plane.

Assume that the identical load (4) with the principal vector

$$\int_L \tau_{nz}(t) ds = 0 \quad (13)$$

is given on the contours of the holes and that there are no stresses acting at infinity. By using the complex potential (11), we arrive at the integral equation (8) with kernel (12), where L is a smooth closed contour. The integral equation for a multiply connected region is constructed with the help of the transition from open cuts in the unbounded elastic plane to closed cuts. We supplement the obtained singular integral equation of the first kind on the closed contour (boundary of the body) with a regularizing term guaranteeing the uniqueness of solution of the integral equation with an arbitrary right-hand side [2, 5, 13]:

$$\frac{1}{d} \int_L \operatorname{Re} \left[\cot \frac{\pi}{d} (t - t') \frac{dt'}{ds'} \right] \gamma'(t) dt + \frac{1}{\pi l} \int_L \gamma'(t) dt = \tau(t'), \quad t' \in L, \quad (14)$$

where l is an arbitrary parameter with dimension of length. Here, the regularizing term on the left-hand side of the equation is equal to zero if condition (13) guaranteeing the uniqueness of displacements is satisfied in traversing the contour of the hole.

If the function $\gamma'(t)$ is found from the integral equation (14), then we get the complex stress potential (11) with the help of which it is possible to determine the level of stresses in the entire body and, in particular, on the contour of the hole [2, 5, 13]:

$$\tau_{sz} = \operatorname{Re} \left[F^+(t) \frac{dt}{ds} \right], \quad t \in L. \quad (15)$$

In more detail, we now consider the problem in the case where the shear stresses $\tau_{yz}^\infty = \tau$ are set at infinity and the contours of the holes are free of external loads:

$$\tau_{nz}^*(t) = 0, \quad t \in L. \quad (16)$$

We seek the complex stress potential in the form

$$F_*(z) = F_0(z) + F(z), \quad (17)$$

where the potential $F_0(z) = -i\tau$ describes the ground stressed state of the continuous plane without holes and the function $F(z)$ determines its perturbed state caused by the holes.

The boundary condition for the perturbed stresses takes the form

$$\tau_{nz}(t) = \tau(t) = -\operatorname{Im} \left[F_0(t) \frac{dt}{ds} \right] = \tau \operatorname{Re} \left[\frac{dt}{ds} \right], \quad t \in L. \quad (18)$$

Then the stresses on the contour of the hole can be found directly in terms of the solution of the integral equation (14). For this purpose, we use the Sochocki–Plemelj formula [14] for integral (11):

$$F_*^+(t') - F_*^-(t') = F^+(t') - F^-(t') = 2\gamma'(t'). \quad (19)$$

Since the region inside the contour L is unloaded, the limit value of the potential $F_*^-(t) = 0$. By using relations (15) and (19), we obtain

$$\tau_{sz}^* = \operatorname{Re} \left[F_*^+(t) \frac{dt}{ds} \right] = 2\gamma'(t) \frac{dt}{ds}, \quad t \in L. \quad (20)$$

To get the numerical solution of Eq. (14) by the method of quadratures [13], we describe the contour L by the parametric equation $t = l\omega(\xi)$, $0 \leq \xi < 2\pi$. By using the quadrature formulas from [2, 13] for the even number of nodes $2N$, we arrive at the following system of algebraic equations;

$$\begin{aligned} \frac{\pi l}{Nd} \operatorname{Re} \left[\frac{\omega'(\eta_m)}{|\omega'(\eta_m)|} \sum_{k=1}^{2N} \gamma'(\xi_k) \cot \frac{\pi l}{d} (\omega(\xi_k) - \omega(\eta_m)) \right] \\ + \frac{1}{N} \sum_{k=1}^{2N} \gamma'(\xi_k) = \operatorname{Re} \left[\frac{\omega'(\eta_m)}{|\omega'(\eta_m)|} \right], \quad m = 1, \dots, 2N, \end{aligned} \quad (21)$$

where

$$\xi_k = \pi(2k-1)/(2N), \quad \eta_m = \pi(m-1)/N, \quad \text{and} \quad \gamma'(\xi_k) = \omega'(\xi_k) \gamma'(l\omega(\xi_k))/\tau.$$

If we determine the quantities $\gamma'(\xi_k)$ from system (21), then we get the stresses $\tau_{sz}^*(t)$ acting on the contour L in the following form:

$$\tau_{sz}^*(\xi_k) = \tau_{sz}^*(l\omega(\xi_k)) = \frac{2\tau\gamma'(\xi_k)}{|\omega'(\xi_k)|}. \quad (22)$$

Note that relation (22) was earlier presented in [2, 5, 12] with certain misprints.

Numerical Results

Consider a periodic system of elliptic and oval holes. For an elliptic hole, the parametric equation of the contour L has the form

$$t = l\omega(\xi) = l(\cos \xi - i\sqrt{\varepsilon} \sin \xi), \quad 0 \leq \xi \leq 2\pi, \quad (23)$$

where $\varepsilon = \rho/l$ is the relative radius of curvature at the vertex of the ellipse and l is the major semiaxis of the ellipse.

The smooth contour L of an oval hole is formed by two symmetric circular arcs of radius R whose ends are smoothly connected with each other by circular arcs of radius ρ . If $\rho = 0$, then the arcs of a circle with radius R cross each other at the angle 2β . The tips of the two-angle hole thus constructed are rounded by arcs of radius ρ . The parametric equation of a part of the contour of the oval hole L in the fourth quarter of the Cartesian coordinate system takes the form

$$t = l\omega_0(\xi) = l \begin{cases} (1-\varepsilon) + \varepsilon \left(\cos \left(\frac{c\xi}{\varepsilon} \right) - i \sin \left(\frac{c\xi}{\varepsilon} \right) \right), & 0 \leq \xi < \xi_B, \\ \varepsilon_R \cos \left[\frac{c(\xi - \xi_B) + \varepsilon_R \theta_B}{\varepsilon_R} \right] \\ + i\varepsilon_R \left[\cos \beta - \sin \left[\frac{c(\xi - \xi_B) + \varepsilon_R \theta_B}{\varepsilon_R} \right] \right], & \xi_B \leq \xi \leq \frac{\pi}{2}, \end{cases} \quad (24)$$

where

$$c = \frac{2\varepsilon\theta_B}{\pi} + \varepsilon_R \left(1 - \frac{2\theta_B}{\pi} \right), \quad \varepsilon_R = \frac{\varepsilon + \sqrt{\varepsilon^2 + (1-2\varepsilon)\sin^2 \beta}}{\sin^2 \beta};$$

$$\theta_B = \arctan \left[\frac{\varepsilon_R \cos \beta}{1-\varepsilon} \right]; \quad \xi_B = \frac{\varepsilon\theta_B}{c};$$

$$\varepsilon = \frac{\rho}{l} \quad \text{and} \quad \varepsilon_R = \frac{R}{l}.$$

This equation is sufficient for the complete description of the contour L because the contour is symmetric about the coordinate axes.

The results of evaluation of the maximal stresses $\tau_{\max} = \tau_{sz}^*(0)$ on the contour of the elliptic hole for small ratios $2a/d$ show that the functions $(1-\gamma)\tau_{\max}/\tau$ tend to certain limit values as the parameter $\gamma = 2l/d \rightarrow 1$ (Fig. 2). Similar dependences were also obtained for a periodic system of oval holes. This enables us to perform the limit transition to two-sided semiinfinite parabolic and V-shaped notches in the elastic plane by assuming that the product $\tau d = T$ remains constant as the ratio $2a/d \rightarrow 0$.

We now compare (see Fig. 3) the stress concentration factors $k_t = \tau_{\max}/\tau$ at the tips of two-sided parabolic and U-shaped notches in a cylindrical body deformed at infinity by shear stresses with the principal vector T with

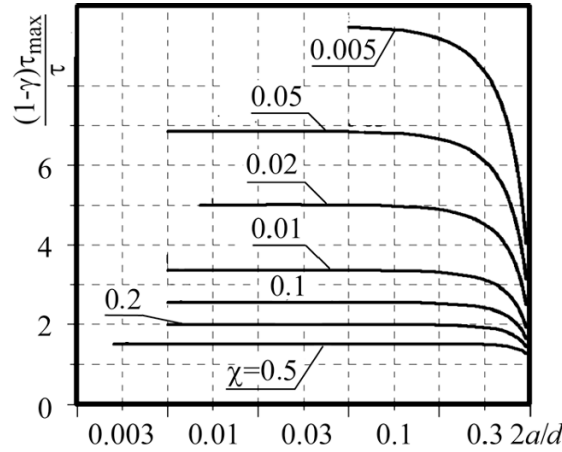


Fig. 2. Dependences of function $(1-\gamma)\tau_{\max}/\tau$ at the tip A of an elliptic hole on the ratio $2a/d$ for various values of the parameter $\chi = \rho/a$.

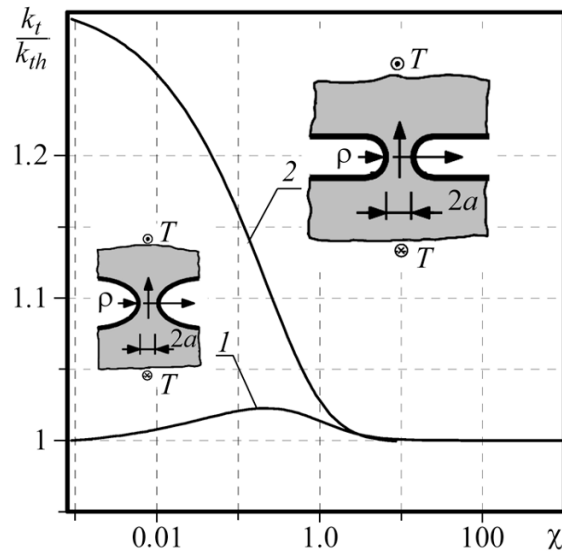


Fig. 3. Dependences of the ratios of stress concentration factors for two-sided parabolic (1) and U-shaped (2) notches to the stress concentration factor for a hyperbolic notch k_{th} on the parameter χ .

the exact analytic solution [15]

$$k_{th} = \frac{1}{\sqrt{\chi} \arctan(1/\sqrt{\chi})}$$

constructed for a two-sided hyperbolic notch. The analysis of these results shows that, for low values of the parameter $\chi = \rho/a$, the shape of the neighborhood of the notch tip under the conditions of antiplane deformation affects the stress concentration much stronger than for the mode I deformation [2, 5]. In the former case, the maximum difference between the stress concentration factors for U-like, parabolic, and hyperbolic notches whose tips have the same radius of curvature can be as large as 30% (Fig. 3). At the same time, it is smaller than 6% for the mode I deformation.

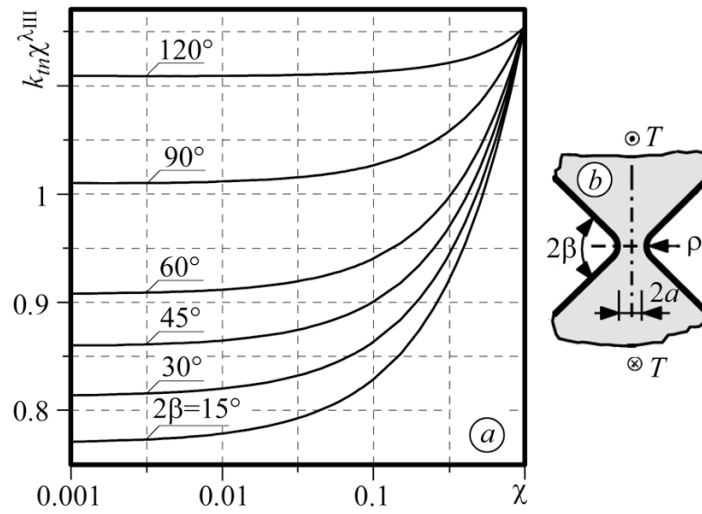


Fig. 4. Dependences of the function $k_m \chi^{\lambda_{III}}$ on the parameter χ for various apex angles 2β (a) of the two-sided rounded V-shaped notch in the elastic plane (b).

Similarly, by using the numerical solution of the problem for a periodic system of closely located oval holes, we obtain the stress concentration factor

$$k_m(\chi) = \frac{\tau_{\max}}{T/(2a)}$$

for the two-sided V-shaped notch with rounded tips in the elastic plane subjected to shear at infinity caused by forces with the principal vector T (Fig. 4). As the parameter χ approaches zero, the product $k_m \chi^{\lambda_{III}}$ (λ_{III} is the parameter of singularity of stresses at the tip of a sharp V- notch) approaches certain asymptotic values corresponding to the sharp two-sided V- notch.

In view of the relation between the stress concentration factor at the tip of a rounded V- notch and the stress intensity factor K_{III}^V at the tip of the corresponding sharp notch [2, 5]

$$K_{III}^V = \frac{\sqrt{2\pi}}{R_{III}} \lim_{\rho \rightarrow 0} \tau_{\max} \rho^{\lambda_{III}} = F_{III}^V \frac{1}{\sqrt{\pi}} \frac{T}{a^{1-\lambda_{III}}}, \quad (25)$$

where

$$F_{III}^V = \frac{\pi}{\sqrt{2}R_{III}} \lim_{\chi \rightarrow 0} k_m \chi^{\lambda_{III}}, \quad \gamma = \frac{\pi}{2} - \beta, \quad \lambda_{III} = 1 - \frac{\pi}{2\pi - 2\beta},$$

and

$$R_{III} = \frac{1 + 17.845\gamma + 20.266\gamma^2 - 19.123\gamma^3 + 9.502\gamma^4 - 1.916\gamma^5}{1 + 14.248\gamma},$$

we plot the dependence of the relative stress intensity factor $F_{III}^V = K_{III}^V \sqrt{\pi} a^{1-\lambda_{III}} / T$ on the apex angle 2β of a two-sided V- notch in a cylindrical body loaded at infinity by the shear forces T (Fig. 5).

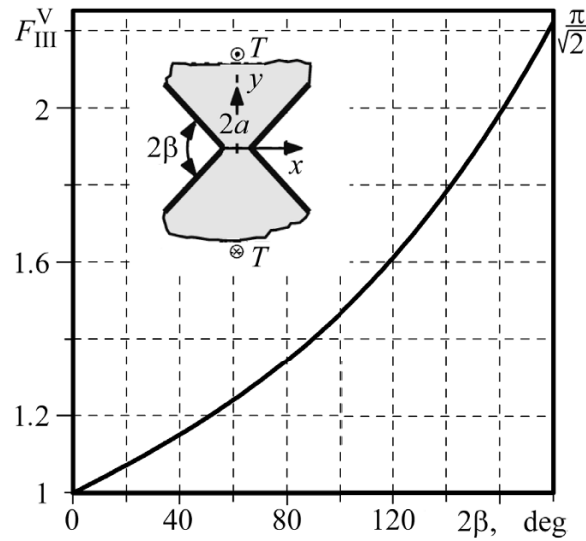


Fig. 5. Dependence of the relative stress intensity factor $F_{III}^V = K_{III}^V \sqrt{\pi a}^{1-\lambda_{III}} / T$ for a two-sided V- notch on the apex angle 2β .

To describe the dependence of the relative stress intensity factor on the angle β , we use the following approximate relation:

$$F_{III}^V = 1 + 0.402\beta + 0.041\beta^2 + 0.126\beta^3,$$

where the relative error does not exceed 0.74% within the interval $\beta \in [0, \pi/2]$.

The presented relations for the stress concentration and stress intensity factors near external two-sided rounded and sharp notches can be used for the asymptotic evaluation of the corresponding parameters for deep two-sided notches in bounded bodies.

CONCLUSIONS

We construct the integral equation for the antiplane periodic problem of the theory of elasticity for an isotropic plane with curvilinear holes. The numerical solution of the problem is presented for closely located elliptic and oval holes. On this basis, we determine the stress concentration factors for two-sided semiinfinite parabolic, U-like, and V-shaped rounded holes in the elastic plane. With the help of the limit transition, as the radius of curvature of the notch approaches zero, the stress intensity factors are computed for the two-sided sharp V-notches with various apex angles.

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