

ENERGY APPROACH TO THE EVALUATION OF THE LIMITING EQUILIBRIUM STATE OF ORTHOTROPIC CRACKED BODIES

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UDC 620.193.01:539.379

We propose an energy criterion of crack propagation in anisotropic bodies based on the equation of energy balance at the onset of crack initiation. In this case, unlike the other well-known energy approaches, parallel with the equation specifying the limiting equilibrium of an anisotropic cracked body, we also use an additional equation for finding the direction of initial crack growth. The formulated criterion is applied to establish the limiting equilibrium state of an infinite orthotropic plate containing a rectilinear crack.

The analysis of the regularities of fracture of reinforced anisotropic materials (including orthotropic composites) is an urgent problem in the development of new structural materials with prescribed properties.

In what follows, we formulate an energy criterion of crack growth in orthotropic bodies based on the equation of energy balance at the onset of crack propagation. Unlike the existing criteria [1], in this case, parallel with the equation specifying the limiting equilibrium of anisotropic cracked bodies, we use the equation for the determination of the initial direction of crack propagation. This equation follows from the hypothesis that the initial direction of crack propagation coincides with the direction in which the ratio of the potential energy to the fracture energy in the elementary volume near the crack tip is maximum.

Computational Model of Crack Growth in Anisotropic Bodies

Consider an elastic anisotropic plane Ω bounded by a line D and containing a crack of length L . According to the law of conservation of energy, the work A of the surface forces in the line D and bulk forces in the plane Ω per unit time is equal to the sum of the growth rate of energy dissipated in the region of plastic deformations $\dot{W}$ and energy $\dot{\Gamma}$ spent per unit time to increase the crack length by ΔL , namely,

$$\dot{A} = \dot{W} + \dot{\Gamma}. \quad (1)$$

The overdot denotes the total derivative of the corresponding quantity with respect to time t .

The crack tip moves according to the law $L = L(t)$, where

$$\frac{dL(t)}{dt} > 0.$$

Since the quantities A , W , and Γ are functions of L and t and

$$\frac{\partial L}{\partial t} \neq 0,$$

Karpenko Physicomechanical Institute, Ukrainian Academy of Sciences, Lviv. Translated from *Fizyko-Khimichna Mekhanika Materialiv*, Vol. 40, No. 5, pp. 53–56, September–October, 2004. Original article submitted August 12, 2004.

we have

$$\frac{\partial(A - W)}{\partial L} = \frac{\partial \Gamma}{\partial L}, \quad (2)$$

By using the definition of the derivative, we rewrite Eq. (2) in the form

$$\lim_{\Delta L \rightarrow 0} \left[\frac{A(L + \Delta L) - W(L + \Delta L) - A(L) + W(L)}{\Delta L} \right] = \lim_{\Delta L \rightarrow 0} \left[\frac{\Gamma(L + \Delta L) - \Gamma(L)}{\Delta L} \right]. \quad (3)$$

Since

$$\lim_{\Delta L \rightarrow 0} \left[\frac{\Gamma(L + \Delta L) - \Gamma(L)}{\Delta L} \right] = \gamma_c,$$

where γ_c is the specific fracture energy spent to form a crack of unit area, and

$$\lim_{\Delta L \rightarrow 0} \left[\frac{A(L + \Delta L) - A(L)}{\Delta L} + \frac{W(L) - W(L + \Delta L)}{\Delta L} \right] = \gamma_s,$$

where $\gamma_s = \delta_I \sigma_0 + \delta_{II} \tau_0$ is the specific plastic strain energy, σ_0 and τ_0 are, respectively, the mean stresses in the process zone near the tips of tensile and shear cracks according to the δ_c -model [1], and δ_I and δ_{II} are, respectively, the crack-tip opening displacements for tensile and shear cracks, we conclude that $\gamma_s = \gamma_c$.

We now express the crack-tip opening displacements via the corresponding stress intensity factors (SIF) [1] and, finally, obtain

$$\gamma_c(\beta) E(\beta) = K_{I\beta}^2(\beta) + K_{II\beta}^2(\beta), \quad (4)$$

where β is the angle of crack propagation relative to its initial axis and $E(\beta)$ is the modulus of elasticity along the line of crack propagation. The SIF admit the following representation:

$$K_{I\beta}(\beta) = p F_I(\beta, L) \quad \text{and} \quad K_{II\beta}(\beta) = p F_{II}(\beta, L). \quad (5)$$

Relations (4) and (5) now imply that

$$p = \sqrt{\frac{E(\beta) \gamma_c(\beta)}{F_I^2(\beta, L) + F_{II}^2(\beta, L)}}. \quad (6)$$

The limiting load p_* can be found from the condition

$$p_* = \sqrt{\Phi(\beta_*)}, \quad \frac{\partial}{\partial \beta} [\Phi(\beta)]|_{\beta=\beta_*} = 0, \quad (7)$$

where

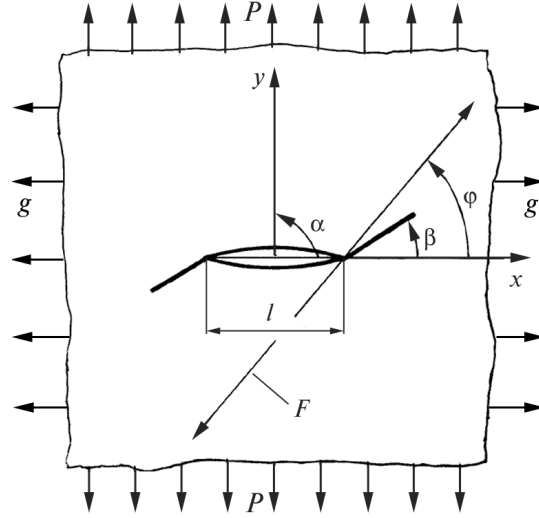


Fig. 1. Schematic diagram of loading of a plate (F denotes a fiber).

$$\Phi(\beta) = \frac{E(\beta)\gamma_c(\beta)}{F_I^2(\beta, L) + F_{II}^2(\beta, L)}$$

and β_* is the angle specifying the initial direction of crack growth.

Limiting Equilibrium State of an Infinite Orthotropic Plate Containing a Rectilinear Crack

Consider an infinite orthotropic plate containing a crack of length l directed at an angle φ to the principal axis of orthotropy of the material. The plate is subjected to the action of uniformly distributed forces p (at an angle α to the axis of the crack) and forces $g = \eta p$ perpendicular to p (Fig.1). The expressions for the SIF $K_{I\beta}$ and $K_{II\beta}$ take the form [2]

$$K_{I\beta} = \sqrt{\frac{\pi l}{2}} \operatorname{Re} \left\{ \frac{1}{\mu_1 - \mu_2} \left[-(\cos \beta + \mu_1 \sin \beta)^{3/2} (\mu_2 \sigma + \tau) + (\cos \beta + \mu_2 \sin \beta)^{3/2} (\mu_1 \sigma + \tau) \right] \right\}, \quad (8)$$

$$K_{II\beta} = \sqrt{\frac{\pi l}{2}} \operatorname{Re} \left\{ \frac{1}{\mu_1 - \mu_2} \left[-(\sin \beta - \mu_1 \cos \beta) \sqrt{\cos \beta + \mu_1 \sin \beta} (\mu_2 \sigma + \tau) + (\sin \beta - \mu_2 \cos \beta) \sqrt{\cos \beta + \mu_2 \sin \beta} (\mu_1 \sigma + \tau) \right] \right\}, \quad (9)$$

where $\sigma = p \sin^2 \alpha + g \cos^2 \alpha$ and $\tau = 0.5(p - g) \sin 2\alpha$ are, respectively, the normal and tangential stresses along the crack line corresponding to the symmetric and skew-symmetric stressed states. The anisotropy of the material is determined by the complex parameters μ_1 and μ_2 defined as the roots of the characteristic equation of the corresponding problem of the theory of elasticity and depending on the orientation of the chosen coordinate system.

The plate of an orthotropic composite is cut out in the direction parallel to the plane of elastic symmetry and

has two mutually perpendicular axes of symmetry (Ox and Oy). We denote the moduli of elasticity along these axes by E_0 and E_{90} ($E_{90} > E_0$), respectively.

If the axis of the crack coincides with the Ox -axis, then

$$\mu_{1,2} = i\sqrt{1 - 2\varepsilon_2 + 0.5\varepsilon_1 \pm \sqrt{2\varepsilon_1 - 4\varepsilon_2 + 4\varepsilon_2^2 + 0.25\varepsilon_1^2 - 2\varepsilon_1\varepsilon_2}}, \quad (10)$$

where

$$\varepsilon_1 = 1 - \frac{E_0}{E_{90}}, \quad \varepsilon_2 = 1 - \frac{E_0}{E_{45}},$$

and E_{45} is the modulus of elasticity along the line making an angle of 45° with the Ox -axis. If the crack is oriented at an arbitrary angle φ to the Ox -axis, then the parameters $\mu_i(\varphi)$ can be found from the following equation [2]:

$$\mu_i(\varphi) = \mu_i \cos \varphi - \frac{\sin \varphi}{\cos \varphi} - \mu_i \sin \varphi \quad (i = 1, 2). \quad (11)$$

Thus, for the plate with rectilinear crack, the problem of determination of the limiting state reduces to the solution of the following equations:

$$\left. \frac{\partial}{\partial \beta} \left[\frac{E(\beta)[169 + 0.211f(\beta + \varphi)]}{F_I^2(\beta, \varphi) + F_{II}^2(\beta, \varphi)} \right] \right|_{\beta=\beta_*} = 0, \quad (12)$$

$$F_I = \sqrt{\frac{l\pi}{2}} \operatorname{Re} \{ (\mu_1 - \mu_2)^{-1} [-(\cos \beta + \mu_1 \sin \beta)^{3/2} (\mu_2 \sigma + \tau) + (\cos \beta + \mu_2 \sin \beta)^{3/2} (\mu_1 \sigma + \tau)] \}, \quad (13)$$

$$F_{II} = \sqrt{\frac{l\pi}{2}} \operatorname{Re} \{ (\mu_1 - \mu_2)^{-1} [-(\sin \beta - \mu_1 \cos \beta) \sqrt{\cos \beta + \mu_1 \sin \beta} (\mu_2 \sigma + \tau) + (\sin \beta - \mu_2 \cos \beta) \sqrt{\cos \beta + \mu_2 \sin \beta} (\mu_1 \sigma + \tau)] \}. \quad (14)$$

In view of relations (13) and (14), we can solve Eq. (12) numerically and construct the dependence of the angle of initial crack growth $\beta_* = \beta_*(\varphi)$ on the angle φ between the crack and the principal axis of orthotropy.

The limiting load p_* is given by the formulas

$$p_* = \sqrt{\frac{E(\beta)[169 + 0.211f(\beta + \varphi)]}{G_I^2(\beta, \varphi) + G_{II}^2(\beta, \varphi)}} \bigg|_{\beta=\beta_*}, \quad (15)$$

$$G_I = \sqrt{\frac{l\pi}{2}} \operatorname{Re} \{ (\mu_1 - \mu_2)^{-1} [-(\cos \beta + \mu_1 \sin \beta)^{3/2} (\mu_2 \sin^2 \alpha + 0.5 \sin 2\alpha) + (\cos \beta + \mu_2 \sin \beta)^{3/2} (\mu_1 \sin^2 \alpha + 0.5 \sin 2\alpha)] \},$$

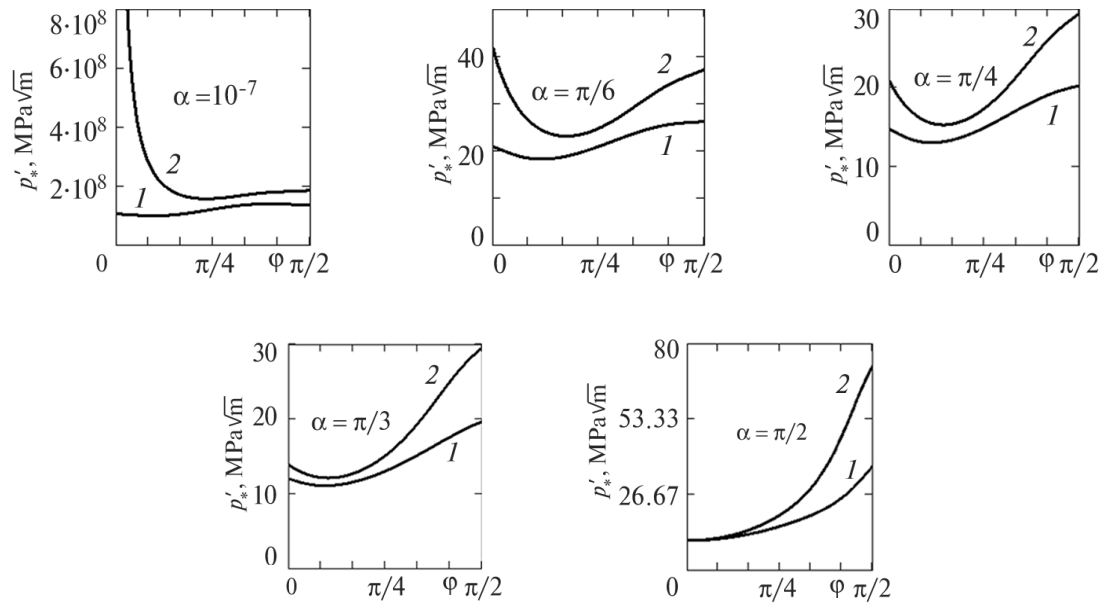


Fig. 2. Dependences of the limiting load $p'_* = p_*\sqrt{l}$ in the ÉF 32-301 material on the direction of the principal axes of orthotropy for different angles of orientation of the load computed according to the energy [relation (16) (curve 1)] and force [5] (curve 2) approaches.

Table 1

φ , rad	$K_{Ic} \text{ (MPa} \cdot \sqrt{\text{m}}) / \gamma_c \text{ (J/m}^2\text{)}$
0	13.0/169
$\pi/6$	16.7/220
$\pi/4$	18.6/272
$5\pi/12$	22.7/366
$\pi/2$	25.5/380

$$G_{II} = \sqrt{\frac{l\pi}{2}} \operatorname{Re}\{(\mu_1 - \mu_2)^{-1} [-(\sin\beta - \mu_1 \cos\beta)\sqrt{\cos\beta + \mu_1 \sin\beta}(\mu_2 \sin^2 \alpha + 0.5 \sin 2\alpha) + (\sin\beta - \mu_2 \cos\beta)\sqrt{\cos\beta + \mu_2 \sin\beta}(\mu_1 \sin^2 \alpha + 0.5 \sin 2\alpha)] \},$$

Examples of Numerical Analysis for Materials with Weak Anisotropy

Assume that an infinite orthotropic plate containing a crack of length l whose axis coincides with the axis of symmetry of the elastic characteristics (Ox) is stretched at infinity by stresses p applied at an angle α to the

axis of the crack. Our aim is to establish the dependence of the limiting load on its orientation relative to the principal axis of anisotropy and the principal axis of the crack.

The numerical analysis was carried out for the ÉF 32-301 composite material (epoxy-phenolic resin reinforced with unidirectional glass fibers) [3] with $E_0 = 21$ GPa, $E_{90} = 33$ GPa, and $E_{45} = 17$ GPa (Fig. 2). The mean values of K_{Ic} [3] and the fracture energy γ_c (computed according to the procedure proposed in [4]) are presented in Table 1 depending on the angle between the crack and the principal axis of orthotropy (corresponding to the maximum modulus of elasticity). The maximum spread of the critical loads is attained as $\alpha \rightarrow \pi/2$ and $\varphi \rightarrow \pi/2$.

CONCLUSIONS

We propose a computational model for the determination of the limiting equilibrium state of anisotropic cracked plates based on the equation of balance of strain and fracture energies of the material. The comparison of the solutions of the problem of limiting equilibrium state of an infinite plate containing a rectilinear crack in tension obtained by the well-known force method [5] and the proposed energy approach shows that the critical values of the load computed by using the energy approach are lower than the values obtained by the force method.

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