



SHORT COMMUNICATIONS

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COMPUTING THE DEFINITE INTEGRALS OF THE
WIENER–HOPF METHOD BY SUMMING POWERS
OF RESIDUES†

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Residue summation is used to compute the (parameter-dependent) definite integrals in the exact solutions of various linear boundary-value problems obtained by the Wiener–Hopf method. Using two specific multidimensional examples, the Navier–Stokes problem (eddy flows in the marginal zone of an ice field) and the Klein–Gordon problem (edge waves under cracked ice), solved here for the first time, it is shown how the typical computational problems of the approach are overcome. The comparative efficiency and economy of the method are demonstrated.

1. This paper is concerned with the problems that arise when computing the definite parametric integrals which arise when solving a number of linear boundary-value problems by the Wiener–Hopf method [1]. These problems arise, in particular, in diffraction theory [2–5], the theory of cracks [6], and the mechanics of a viscous fluid [7]. The integrals have the form

$$\Psi(x, z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{F(z, k)}{\lambda(z, k)} \exp(ikx) dk, \quad (1)$$

where $F(z, k)$ and $\lambda(x, k)$ are integral analytic functions of z and k . The existence of the integral (1) is guaranteed for these functions, the only zeros of $\lambda(x, k)$ lie in one of the half-planes of the complex variable k , and x is a real parameter.

The integrals in (1) will be calculated by the residue summation method

$$\Psi(x, z) = \sum_{n=1}^{\infty} F(z, k_n) \exp(ik_n x) R_n, \quad (2)$$

$$R_n = \text{Res}(1/\lambda)|_{k=k_n}$$

Here k_n , $n = 1, 2, \dots$, are the roots of the function $\lambda(x, k)$ in order of their modulus and lie in the upper and lower half-planes of k .

The method of computing integrals (1) currently used goes back as far as [8]. It is based on the choice of a special simplified asymptotically compatible fractional-polynomial approximation for certain functions λ (otherwise in some problems their exact expression will contain infinite products) and then applying the usual quadrature method [9] to (1). This approach has certain limitations and is only successful in the important but simple case when $F = \text{const}$, x is fixed, and λ takes certain specific

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forms, or in finding asymptotic forms of the solutions [6]. Of course, in many cases it is possible to avoid having to use the Wiener–Hopf method by solving the initial linear systems of equations by numerical methods. However, such an approach often requires a very large volume of computations to obtain satisfactory accuracy, and this is only possible using large computers.

Our purpose here is to use the example of two new problems to identify typical difficulties in applying (2) and ways of overcoming them, to demonstrate the efficiency of the method and thus draw attention to further applications in this area.

2. Vortical flows of a liquid in a marginal ice zone. We will consider the formation of wind sea flows in the marginal zone of an ice mantle (see Fig. 1, in which the upward arrow indicates the wind direction). The problem was solved analytically for the first time in [7], and the first qualitative results of numerical investigations are given in [10]. The ice mantle was modelled by a solid semi-infinite cover on the surface of a liquid layer in dimensionless variables ($z = 1, x \geq 0$) (cf. Fig. 1). The liquid was caused to move by steady wind stresses on the semi-infinite free surface ($z = 1, x \leq 0$). Sticking conditions were satisfied on the surface of the ice mantle and at the bottom ($z = 0$). The problem was solved in the framework of the stationary two-dimensional linearized system of Navier–Stokes equations. Solution of the flow equations reduced to solving the biharmonic equation for the stream function Ψ

$$\Delta\Delta\Psi = 0, \quad 0 \leq z \leq 1, \quad -\infty < x < +\infty,$$

with boundary conditions

$$\Psi_x = \Psi_z = 0, \quad z = 0, \quad -\infty < x < +\infty,$$

$$\Psi_x = \Psi_z = 0, \quad z = 1, \quad x \geq 0,$$

$$\Psi_x = 0, \quad \Psi_{xz} = C_w |V_w| V_w, \quad z = 1, \quad x \leq 0,$$

where C_w is the coefficient of wind friction on the surface of the liquid, V_w is the wind speed, and Ψ is assumed to decay under the ice: $\Psi(+\infty, z) = 0$.

The solution of the system obtained by the Wiener–Hopf method can be represented in the form of combinations of integrals of type (1), where

$$F(z, k) = (1 - z) \operatorname{sh} k \operatorname{sh} (kz) - kz \operatorname{sh} [k(1 - z)], \quad (3)$$

$$\lambda(x, k) = \begin{cases} k(k^2 - \operatorname{sh}^2 k) \lambda^+(k), & x \geq 0, \\ k^2 [2k - \operatorname{sh}(2k)] / \lambda^-(k), & x \leq 0, \end{cases}$$

$$\lambda^+(k) \lambda^-(k) = k [\operatorname{sh}(2k) - 2k] / (\operatorname{sh}^2 k - k^2).$$

The functions λ^+ and λ^- are written as infinite products which converge absolutely outside their singularities

$$\lambda^{\pm}(k) = \exp [(1 \mp 2ik/\pi) \ln 2] \prod_{n=1}^{\infty} \frac{(1 \pm 2k/\zeta_n)}{(1 \pm k/\zeta_n)(1 \pm k/\xi_n)}, \quad (4)$$

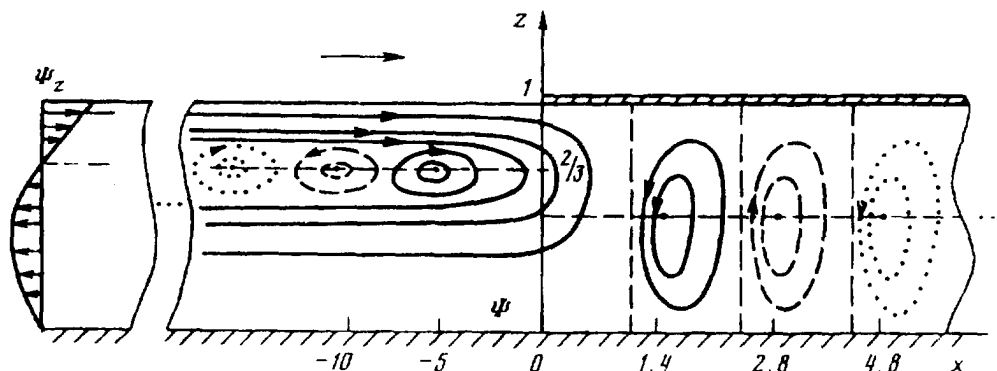


Fig. 1.

where ζ_n and ξ_n are the roots of the equations $k = \text{sh } k$ and $k = -\text{sh } k$, respectively, $\text{Im } \zeta_n > 0$, $\text{Im } \xi_n > 0$.

Apart from known terms, $\Psi(x, z)$ is the stream function for the flow of the liquid. Together with its first derivatives with respect to x and z , it is continuous over the whole region of motion including the point $x = 0, z = 1$.

In order to construct a picture of the streamlines in the flow and investigate the fine vortex structure, (1) must be computed with respect to (2) for several values of x and z . The summation consists of three stages: computing roots of the equation $\lambda(x, k) = 0$, computing the residues R_n , and finding partial sums of the series (2) which give satisfactory accuracy for Ψ .

Computing the roots of $\lambda = 0$ in the case (3) reduces to solving

$$\text{sh } k - \alpha k = 0, \quad \alpha = \pm 1, \quad (5)$$

by Newton's iterative method with the initial approximation for the n th root k_n^0 computed with the formula

$$k_n^0 = \ln [(4n + \alpha)\pi] + i\alpha(4n + \alpha)\pi/2, \quad (6)$$

$n = 0, 1, \dots$, for $\alpha = +1$ and $n = 1, 2, \dots$, for $\alpha = -1$. Formula (6) is an asymptotic expression for the roots of (5) as $|k| \rightarrow +\infty$. After eight iterations, the discrepancy in the equation was not more than 8×10^{-10} for the first 4000 roots, all the computations were done to double accuracy (16 decimal places) on an RDR mini-computer at 40 thousand operations per second.

The residues were then computed using the partial products (4), where the number of factors in the products were taken equal to 1250, 2250 and 3250 for monitoring purposes.

The values of (2) were calculated with acceptable accuracy with the first 30 residues or terms of (2), but an additional problem arose in calculating the partial sums (2) due to the weak convergence of the series for small x . (For large x , the series (2) is majorized by the series $\sum_n \exp(-2\pi n|x|)$ and its slow convergence does not cause any difficulty.) The convergence improved significantly when summation of the arithmetic means or Cesaro's method was used [11]. The use of this method provided acceptable accuracy for a small number of terms of the series. The error $\delta\Psi$ was estimated from the relative difference between the value of Ψ on the right and left of $x = 0$. The first five terms of the series gave an error of 0.5, the first 15 terms gave $\delta\Psi = 0.19$, 30 terms gave $\delta\Psi = 0.08$ in the case where 3250 multipliers were used. Although the results could be improved further, they were adequate for studying the flow structure. Calculations of the flow in a two-dimensional region took tens of minutes.

Our approach involves a fixed number (60 in all) of computations of the unwieldy functions (4), which take up a large part of the computer time. It is extremely uneconomical and impracticable to use quadrature formulae to integrate (1) in this case, because that would require a much larger number of computations of (4) and would depend on the choice of grids and the values of x due to the strongly oscillating coefficient $\exp(ikx)$. Also, in order to compute (4) for large k , due to the increase in the number of terms in the partial products it is necessary to know a greater and greater number of roots. The attempt to use Koiter's approximate solution [8] for the function (4) and then compute the residue sums gave an unsatisfactory error.

Figure 1 shows the scheme obtained for wind flow of the liquid in the marginal zone of an ice mantle (it contains the isolines of Ψ , and the profile on the left is the function Ψ_z as $x \rightarrow -\infty$). The vortex system is clearly seen. The eddies decrease exponentially with distance from $x = 0$. Very fine grids would obviously be required to obtain a similar pattern of flow by solving the original Navier-Stokes equations directly. The coordinates of the centres of the first vortices for $x \geq 0$ are: $z = 0.5$ and $x_1 = 1.4$, $x_2 = 2.8$, $x_3 = 4.2$, for $x \leq 0$: $z = 2/3$ and $x_{-1} = -4.7$, $x_{-2} = -10$. The stream function Ψ is zero for $z = 0, z = 1$, and the local values of its first extrema for $x \geq 0$ are

$$\Psi(1, 0.5) = -1.7 \cdot 10^{-4}, \quad \Psi(2.4, 0.5) = 4.7 \cdot 10^{-7},$$

$$\Psi(3.8, 0.5) = -1.3 \cdot 10^{-9},$$

for $x \leq 0$: $\Psi(-2.23, 0.67) = +7.4 \times 10^{-2}$, and so on.

Some features of the computation of this section are worth noting. Firstly, it is advisable to use generalized summation method to increase the rate of convergence of series (2). Secondly, calculation of the roots of $\lambda(x, k) = 0$ requires an analysis of their location, a "good" initial approximation for the

iteration, and verification of the fact that they lie entirely in the bounded region of k . Whereas this last feature has not given rise to any particular difficulty in this section, it does in the next problem.

3. *Edge waves in a liquid under an elastic sheet with a crack.* We will solve the problem of constructing three-dimensional potential wave motions in a finite layer of an ideal incompressible liquid lying under an elastic (ice) sheet with a rectilinear through-crack. We will be interested in the question of the conditions of existence, dispersion relations, profiles and forms of edge waves propagating without decay along the crack, but decaying further away. The mathematical apparatus needed to solve problems of this kind was formulated in [6], where analytic solutions were obtained to describe the diffraction of plane waves at a crack in an ice mantle floating on the surface of a liquid. However, the question of the existence of edge waves was left unanswered and is answered here for the first time.

In dimensionless form, the problem reduces to finding a non-trivial solution of the Klein–Gordon equation for the velocity potential of a liquid (cf. [6]):

$$\mu \Psi_{xx} + \Psi_{zz} - \mu \Psi = 0, \quad -1 \leq z \leq 0, \quad -\infty < x < +\infty,$$

with boundary conditions

$$-\gamma \Psi + \left[1 + D\mu^2 \left(-1 + \frac{\partial^2}{\partial x^2} \right)^2 \right] \Psi_z = 0, \quad z = 0, \quad -\infty < x < +\infty,$$

$$\Psi_z = 0, \quad z = -1, \quad -\infty < x < +\infty,$$

$$\mu = (HK_x)^2, \quad \gamma = \omega^2 H/g, \quad D = Eh^3/[\rho g(1 - \nu^2)H^4].$$

Here z and x are the vertical and horizontal coordinates, H is the depth of the liquid, K_x is the wave number along the crack with coordinates $x = z = 0$, ω is the wave frequency, E is Young's modulus, ν is Poisson's ratio, h is the thickness of the sheet, ρ is the density of the liquid, and g is the acceleration due to gravity.

The solution must also satisfy contact boundary conditions, including the absence of shear forces and bending moments on the sides of the crack:

$$\lim_{x \rightarrow \pm 0} \left(\frac{\partial^2}{\partial x^2} - \nu \right) \Psi_z \Big|_{z=0} = \lim_{x \rightarrow \pm 0} \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial x^2} - \nu \right) \Psi_z \Big|_{z=0} = 0. \quad (7)$$

The flow velocity potential Ψ is determined by solving the Klein–Gordon equation by the Wiener–Hopf method and can be represented, in dimensionless form, in the general form (1) (cf. [6]), where

$$F(z, k) = P_3(k) \operatorname{ch} [\tau(1+z)] / \operatorname{ch} \tau, \quad (8a)$$

$$\lambda(x, k) = \lambda(\tau) = \gamma - (1 + D\tau^4) \tau \operatorname{th} \tau, \quad (8b)$$

$$\tau = [\mu(1 + k^2)]^{1/2},$$

where $P_3(k)$ is a third-degree polynomial in k , and its coefficients are found from (7). We now proceed differently from [6]. Substituting (8) into (7), we obtain a system of linear equations for the four coefficients of the polynomial. For non-trivial solutions the determinant Δ of the system must vanish; Δ is a non-linear combination of integrals of the form (1), and when these are transformed in accordance with (7) as $x \rightarrow \pm 0$ for $z = 0$ they take the slightly simpler form

$$\Psi_n = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} [k^n \operatorname{th} \tau / \lambda(k)] dk, \quad n = 0, 1, 2, 3. \quad (9)$$

The equation $\Delta = 0$ implicitly determines the dispersion curve $\gamma = \gamma(\mu)$ for edge waves. The wave forms and profiles are easy to determine from that curve.

The investigations then continue with numerous computations of the integrals (9) for different combinations of γ and μ , computation of Δ and the determination of γ and μ for which $\Delta = 0$. The computations of (9) were carried out by summation of the residues (2), in three stages: computing the roots of the equation $\lambda(k) = 0$ (or $\lambda(\tau) = 0$), computing the residues R_n and finding the partial sums of (2) to obtain Ψ_n .

Finding the roots of the equations was a rather unusual stage of the process and required special attention. The roots of the equation $\lambda(\tau) = 0$ were found to belong to three different families. The first consisted of two non-zero real roots, the second consisted of a denumerable number of imaginary roots,

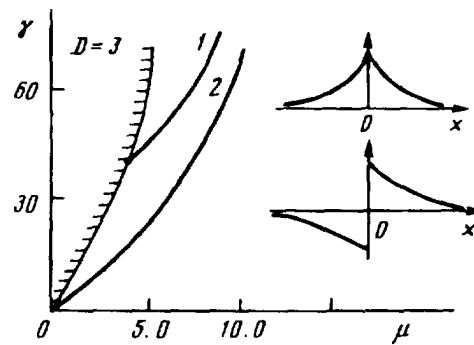


Fig. 2.

and the third consisted of four complex roots not lying on the axes of the complex plane. Each family required the use of Newton's method with its own initial approximation. In particular, almost universal initial approximations were found for all roots of the families. For the first family, we obtained $\pm(\gamma/D)^{1/5}$, for the second, $\pm[\pi(k-1/2)+0.01]i$, where $k=1, 2, \dots$, are the numbers of the roots, and for the third, $\pm 1.01(1 \pm i)/(4D)^{1/4}$. We checked that the roots of different families were different and that the roots in the second family were in strict order. This was done to check that all the roots had been computed, as well as to detect any cases of equal roots (degeneracy of third repetition). In the degenerate case, the corresponding residues become zero. If no allowance is made for this, the dispersion curve behaves anomalously. All the iterations were done to double accuracy, there were not more than 2000 iterations; the absolute discrepancy in the equation for the first 52 roots in the first quadrant of the complex plane, for example, was not more than 4×10^{-14} for any combination of γ and μ .

The second and third stages did not present any difficulty. In particular, only the first 30 terms of the series were needed to obtain acceptable accuracy in (2), correct to the third or fourth decimal place. (It can be shown that the series (2) converges absolutely in this case, and is majorized by the series $\sum_n 1/n^3$.) The computations were carried out with 10, 20, 30, 40 and 50 terms, for monitoring purposes. It took a few minutes to compute one integral.

Our method required up to 50 residues to be computed. The method using quadrature formulae could be applied in this case, but, due to its slow convergence and additional complications associated with the need to allow for singularities in the integrand on the real axis, involves a much larger amount of computation. Koiter's method cannot be used for this problem. Also, it is impossible to solve the original boundary equations directly, even in the part where the dispersion curve has been found.

Figure 2 shows the dispersion curves for symmetric and non-symmetric waves (1 — for symmetric and 2 — for non-symmetric edge waves), and two typical wave amplitude profiles on the right. The shaded region shows the "prohibited zone" of the dispersion plane, in which there are no edge wave solutions. Thus, the solution of this problem has yielded a new type of waveguide, not previously described in the literature.

We have given the example of two specific problems to demonstrate an approach to the computation of the integrals in the Wiener-Hopf method which reduces them to the summation of series of residues of the integrands. The typical difficulties that arise using it and methods of overcoming these have been described. The efficiency and speed of the method are confirmed by calculations. It can be used to obtain and investigate exact solutions of the original system of equations, involves very much less computation than other methods, and can be recommended for practical use.

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