

INFLUENCE OF SHEAR LOADING ON THE SHAPES OF OPTIMAL HOLES IN A QUASI-ORTHOTROPIC PLATE

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We develop an analytic-numerical algorithm for the determination of the optimal shape of smooth holes (with minimum stress concentration) in quasiorthotropic plates subjected to biaxial tension and shear at infinity. We solved the inverse problems of the theory of elasticity for quasiorthotropic bodies with unknown boundaries. To do this, we use the effective solutions of the corresponding direct problems obtained by the method of singular integral equations. The solution of the inverse problem with unknown shape of the hole is reduced to the minimization of the multiparameter functional of standard deviations of tensile stresses along the contour of the hole from their averaged values. We determine the shapes of optimal holes in quasiorthotropic plate for different types of biaxial tensions and shears at infinity and the degree of its orthotropy.

Keywords: quasiorthotropic plate, tensile and shear loading, optimal holes, stress concentration, singular integral equations.

Introduction

To establish the optimal shape of structural elements (subjected to the action a given load) and various structural holes in these elements, we solve the inverse problems of the theory of elasticity for the domains with unknown boundaries by minimizing quality functionals of various types (deviations of the contour stresses from the nominal values, variations of the strength of structures, or the increments of energy introduced by the holes in a given homogeneous stress field). The formulation of inverse problems is based on the solutions of the corresponding direct problems for given shapes of the holes. Numerous problems of finding the optimal shapes of holes in elastic isotropic planes were solved by using different methods [1–5]. Under the conditions of pure shear of an isotropic plane at infinity, the problem of minimization of alternating stresses in piecewise smooth holes was investigated in [6, 7].

The problems of optimization of the shapes of holes in anisotropic bodies (of different composite materials) are much more complicated because their stressed state strongly depends both on the loading mode and on the orientation of the holes relative to the axes of anisotropy of the material. It is necessary to determine the shapes of the holes for which the concentration of stresses on their contours is minimum. For anisotropic plates, some solutions of problems of this kind were obtained in [4, 8].

Holes of the optimal shape in quasiorthotropic plates were investigated under the action of biaxial tension along the axes of orthotropy [9]. In what follows, this approach is used to study the general case of homogeneous stressed state at infinity, which enables us to determine the influence of additional shear loading upon the stress-strain state of a quasiorthotropic plate with the hole.

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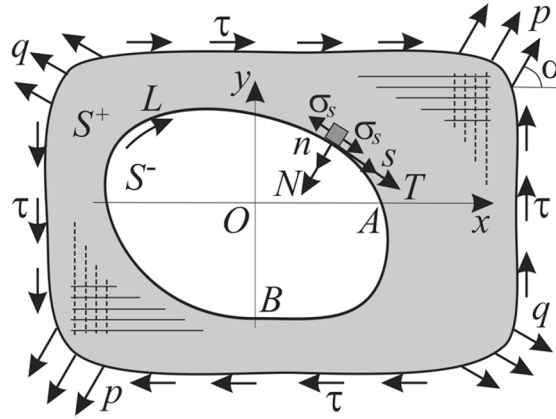


Fig. 1. Biaxial tension and shear of a quasiorthotropic plate with smooth hole.

Direct two-dimensional problems of the theory of elasticity for quasiisotropic bodies with smooth holes were solved by the method of singular integral equations (SIE) [10–12]. The solutions of inverse problems in domains with unknown boundaries were constructed by the earlier developed methods [5, 9]. To minimize the multiparameter functional of standard deviations of the contour stresses (tensile stresses acting along the contour in the tangential direction) from their values averaged over the whole contour, we apply the quasi-Newton method with finite-difference first derivatives [13].

Formulation of the Inverse Problem

Consider a two-dimensional problem of the theory of elasticity for an infinite quasiorthotropic plate containing a smooth hole whose contour L is subjected to the action of stresses

$$N^*(t) + iT^*(t) = p^*(t), \quad t \in L, \quad (1)$$

where $N^*(t)$ and $T^*(t)$ are, respectively, the normal and tangential components of the vector of stresses applied on the contour of the hole with outer normal n . As a positive direction, we choose the direction of tracing the contour L for which the domain of the body S^+ remains to the left. At infinity, the elastic plate is subjected to the action of biaxial tension in mutually perpendicular directions by stresses p (directed at an angle α to the axis Ox) and q and shear τ (Fig. 1). The axes of a Cartesian coordinate system xOy are directed along the principal axes of orthotropy of the plate. The material of the plate is assumed to be quasiorthotropic, i.e., a degenerate orthotropic material, for which the roots of the characteristic equations are multiple and pure imaginary $\mu_1 = \mu_2 = i\gamma$ [10, 11]. For the plane stressed state, $\gamma = \sqrt[4]{E_x/E_y}$ is the main parameter of orthotropy and the shear modulus is expressed in terms of the other constants by the formula

$$G = \frac{E_x}{2(\sqrt{E_x/E_y} + \nu_{xy})},$$

where E_x and E_y are the moduli of elasticity along the x - and y -axes, respectively, and ν_{xy} is Poisson's ratio. For an isotropic material, we have $\gamma = 1$.

It is necessary to find the shape of a smooth hole L for which the normal stresses in the direction of tangent take constant values

$$\sigma_s^*(t) = c = \text{const}, \quad t \in L. \quad (2)$$

These holes are called optimal. Under arbitrary loading applied to the contour of the hole (1) and the plate at infinity, the constant c is found from the solution of the problem. For a free hole ($p^*(t) = 0$) or hole loaded by constant normal and shear stresses ($p^*(t) = N_0 + iT_0 = \text{const}$) in the isotropic plate, this constant is known ($c = p + q$ or $c = p + q - N_0$) and is determined by the first invariant of the stress tensor ($N_0 + \sigma_s = p + q$) [1–3, 14].

In order that the solution of inverse problem with unknown boundary of the body be single-valued, it is necessary to impose auxiliary conditions. For a plate containing a single hole, it suffices to specify a single parameter, e.g., the diameter or area of the hole. Moreover, problems of this kind are not always solvable. In what follows, we consider only loads for which holes with the optimal shapes exist.

Direct Problem

We solve the first main problem of the theory of elasticity for a two-dimensional quasiisotropic body with smooth hole by the method of SIE. By the method of superposition, we seek the complex stress potentials for the quasiisotropic body in the form [11]

$$\Phi_1^*(z_1) = \Phi_1^0(z_1) + \Phi_1(z_1), \quad \Psi_1^*(z_1) = \Psi_1^0(z_1) + \Psi_1(z_1), \quad (3)$$

where

$$\Phi_1^0(z_1) = \frac{\sigma_y^\infty + \gamma^{-2}\sigma_x^\infty}{4}, \quad \Psi_1^0(z_1) = \frac{\sigma_y^\infty - \gamma^{-2}\sigma_x^\infty}{2} + \frac{i\tau_{xy}^\infty}{\gamma} \quad (4)$$

are given functions that describe the homogeneous stress-strain state in the solid quasiorthotropic plate without holes and the potentials $\Phi_1(z_1)$ and $\Psi_1(z_1)$ specify the stressed state perturbed by the hole [11, 12]. For the analyzed loading of the plate in relations (4), we find

$$\sigma_x^\infty = p \cos^2 \alpha + q \sin^2 \alpha, \quad \sigma_y^\infty = q \cos^2 \alpha + p \sin^2 \alpha, \quad \tau_{xy}^\infty = \tau + (p - q) \sin \alpha \cos \alpha, \quad (5)$$

where α is the angle between the Ox -axis and the direction of the stress p (Fig. 1). The complex stress potentials $\Phi_1(z_1)$ and $\Psi_1(z_1)$ in expressions (3) and the integral equations for the problem are written in the mathematical plane $z_1 = x + i\gamma y$ and have the form similar to that obtained for the isotropic material [9, 11, 12]. Here and in what follows, subscript “1” means that the corresponding quantities are presented in the plane z_1 .

We represent the boundary condition (1) for the perturbed stressed state of the plate in the plane z_1 in the following form [11]:

$$\left[\left(\frac{i}{\gamma} \right) X_n - Y_n \right] \frac{ds}{dt_1} = \tilde{P}(t) = \tilde{P}_1(t_1) = \frac{1}{2\gamma} \left[(1 + \gamma)p(t) - (1 - \gamma)\overline{p(t)} \frac{d\bar{t}}{dt} \right] \frac{dt}{dt_1}, \quad (6)$$

where

$$t = x + iy \in L, \quad t_1 = x_1 + iy_1 = x + i\gamma y \in L_1,$$

X_n and Y_n are the Cartesian components of the vector of stresses acting from the outer normal n on the contour L (Fig. 1);

$$p(t) = p^*(t) - \{p + q + [2i\tau - \exp(2i\alpha)(p - q)]\overline{dt/dt}\}/2$$

are the perturbed stresses $N(t) + iT(t)$ on the contour L that do not depend on the elastic constants of the material of the plate and remain identical to the corresponding constants for the isotropic body. Finding the limit values of the complex potentials (3) as $z_1 \rightarrow t_1 \in L_1$ and satisfying the boundary condition (6) with the help of these values, we arrive at the SIE [9] whose numerical solution can be found by the method of quadratures [11].

If the contour of the hole is free of loads ($p^*(t) = 0$), then the stresses acting in the inner domain of the hole (S^-) are equal to zero, and, for the corresponding limiting values of potentials (3), we get

$$[\Phi_1^*(t_1)]^- = [\Psi_1^*(t_1)]^- = 0.$$

Then we determine the contour stresses by using the first invariant of the stress tensor according to the formula

$$\sigma_s^*(t) = \sigma_x^{*+}(t_1) + \sigma_y^{*+}(t_1) = \operatorname{Re}\{2(1 + \gamma^2)\Phi_1^{*+}(t_1) + (1 - \gamma^2)[\bar{t}_1\Phi_1^{*+}(t_1) + \Psi_1^{*+}(t_1)]\}.$$

By using the following expression for the unknown function continuous on the contour L_1 :

$$ig_1'(t_1) = [\Phi_1^*(t_1)]^+ - [\Phi_1^*(t_1)]^-,$$

where the superscripts indicate the limit values of quantities as $z_1 \rightarrow t_1$ to the left (+) and to the right (−) with respect to the chosen positive direction, and the dependence

$$[\bar{t}_1\Phi_1^{*+}(t_1) + \Psi_1^{*+}(t_1)]^+ - [\bar{t}_1\Phi_1^{*+}(t_1) + \Psi_1^{*+}(t_1)]^- = i[\overline{g_1'(t_1)} - g_1'(t_1)]d\bar{t}_1/dt_1,$$

we obtain [9]:

$$\sigma_s = 2 \operatorname{Im}\{g_1'(t_1)\} \left[(1 - \gamma^2) \operatorname{Re}\left\{\frac{d\bar{t}_1}{dt_1}\right\} - (1 + \gamma^2) \right]. \quad (7)$$

Here and in what follows, $\sigma_s = \sigma_s^*(t)$, $t \in L$ are the normal tensile stresses on the contour of the hole in the tangential direction (Fig. 1).

Numerical Algorithm for the Determination of the Optimal Shapes of the Holes

On the basis of the numerical solution of the SIE [9] for the initially given circular hole L , we determine stresses (7) at n nodal points of the contour $t = \omega(\xi_j)$, $\xi_j = 2\pi(j-1)/n$, $j = \overline{1, n}$ and construct the functional

of standard deviations from the mean value [5, 9]

$$F[\mathbf{P}] = \frac{1}{n} \sum_{j=1}^n \left\{ \frac{\sigma_s[\omega(\mathbf{P}, \xi_j)] - \tilde{\sigma}_s}{\tilde{\sigma}_s} \right\}^2, \quad \tilde{\sigma}_s = \frac{1}{n} \sum_{j=1}^n \sigma_s(\xi_j), \quad (8)$$

where $\mathbf{P} = (P_1, P_2, \dots, P_M)$ is the set of the dimensionless parameters of the problem of optimization with the help of which we determine the M -parametric equation of the contour L in the form

$$t = \omega(\mathbf{P}, \xi) = x(\mathbf{P}, \xi) - iy(\mathbf{P}, \xi), \quad \xi \in [0; 2\pi]. \quad (9)$$

Minimizing functional (8) with respect to the parameters P_m ($m = \overline{1, M}$), we find their values for which the stresses $\sigma_s(x)$ are equalized over the entire contour L , i.e., the functional condition $F[\mathbf{P}] < \Delta$ is satisfied, where Δ is a given characteristic of admissible deviations of the contour stresses (7) on the required contour from the mean value. To minimize the multiparameter functional $F[\mathbf{P}]$, we used a quasi-Newton method with finite-difference first derivatives [13].

Numerical Results

The algorithm used for the determination of the optimal shape of the hole in the quasiorthotropic plate is presented for a set of values of the orthotropy parameter of the material of the plate $\gamma = \sqrt[4]{E_x/E_y}$ and different levels of biaxial tension (p, q) and shear (τ) of the plate at infinity (Fig. 1). The optimal contours of the holes are sought in the form (9) with the help of trigonometric series with ten parameters P_m :

$$\begin{aligned} \frac{x(\mathbf{P}, \xi)}{a} &= P_1 \cos(\xi) + P_2 \sin(\xi) + P_3 \cos(3\xi) + P_4 \sin(3\xi) \\ &\quad + (1 - P_1 - P_3) \cos(5\xi) + P_5 \sin(5\xi), \\ \frac{y(\mathbf{P}, \xi)}{a} &= P_6 \sin(\xi) + P_7 \cos(\xi) + P_8 \sin(3\xi) + P_9 \cos(3\xi) \\ &\quad + P_{10} \sin(5\xi) - (P_7 + P_9) \cos(5\xi), \quad \xi \in [0; 2\pi], \end{aligned} \quad (10)$$

where we have taken into account the property of symmetry of the problem about rotations by the angle π ($\omega(\xi + \pi) = -\omega(\xi)$, $\xi \in [0; \pi]$) and the fact that the contour is fixed at the point $A(a; 0)$ ($x(\mathbf{P}, 0) = a$, $y(\mathbf{P}, 0) = 0$).

The presence of shear loading ($t = \{0.1\sigma; 0.2\sigma; 0.3\sigma; 0.4\sigma; 0.5\sigma\}$, where σ is the general loading parameter) in the case of uniform tension ($p = q = \sigma$, $\alpha = \pi/2$) of the quasiorthotropic plate leads to a significant elongation of the shapes of optimal holes along the line inclined at an angle $\pi/4$ to the Ox -axis (Fig. 2). For a relatively low level of orthotropy $\gamma = 1.4$, the shapes of the optimal holes are similar to elliptic (Fig. 2a), which are optimal for the isotropic plate [1, 14]. As the orthotropy of the material of the plate ($\gamma = 2.1$; $\gamma = 2.6$) increases, the optimal holes become similar to “curvilinear rectangles” inclined at an angle of $\pi/4$ to the Ox -axis (Figs. 2b, c).

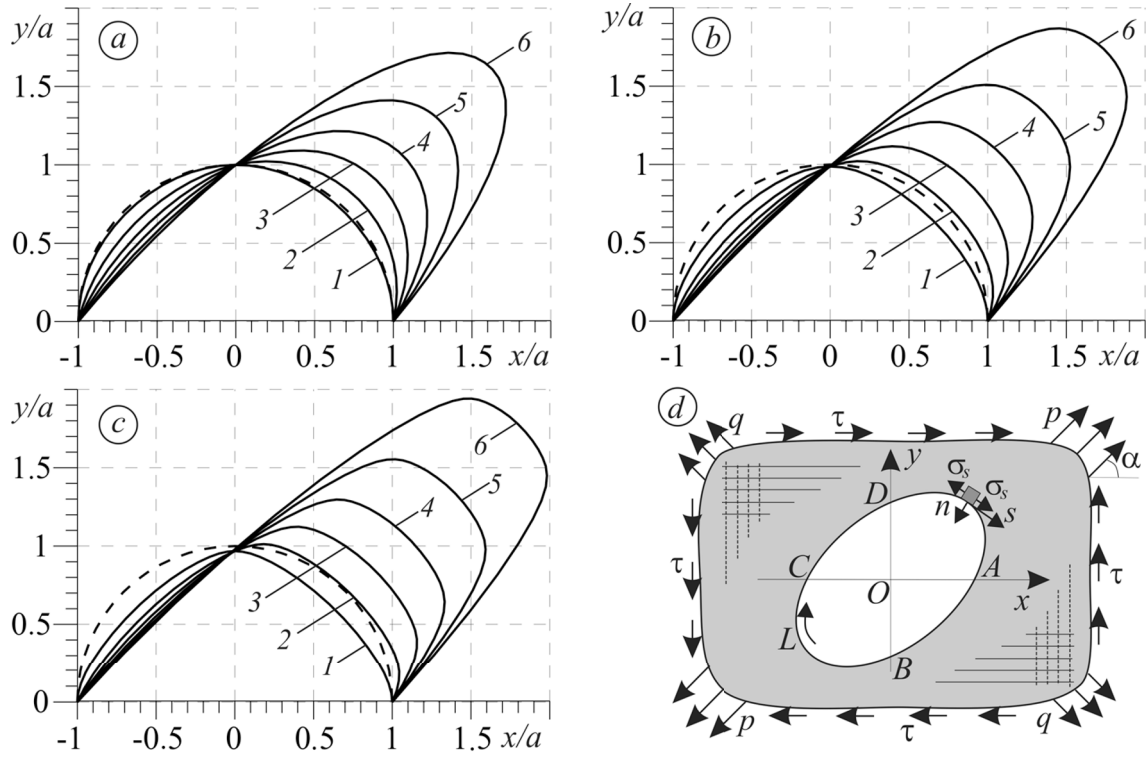


Fig. 2. Influence of additional shear of the quasiorthotropic plate ($\tau = \{0; 0.1; 0.2; 0.3; 0.4; 0.5\}\sigma$ correspond to curves 1–6) in the case of its omnidirectional tension ($p = q = \sigma$) on the shapes of the optimal holes (their upper halves are marked by CDA) as compared with the semicircle (dashed line) for $\gamma = 1.4$ (a), 2.1 (b), and 2.6 (c); (d) schematic diagram of the problem.

The chosen values of the orthotropy parameter $\gamma = \sqrt[4]{E_x/E_y} \in \{1.4; 2.1; 2.6\}$ approximately correspond to the ratios E_x/E_y for orthotropic materials: E-Glass/Epoxy ($E_x = 41.4$ GPa; $E_y = 10.3$ GPa; $G_{xy} = 4.27$ GPa, and $\nu_{xy} = 0.28$); Carbon/Epoxy (IM7/977-3; $E_x = 191$ GPa; $E_y = 9.93$ GPa; $G_{xy} = 7.79$ GPa, and $\nu_{xy} = 0.35$), and Graphite/Epoxy (G_y-70/934; $E_x = 294$ GPa; $E_y = 6.34$ GPa; $G_{xy} = 4.90$ GPa, and $\nu_{xy} = 0.23$) [15].

For the above-mentioned loads, we determined the distributions of the contour stresses (7) on the initial circular and found optimal holes (Fig. 3). As the level of orthotropy of the plate increases from $\gamma = 1.4$ to 2.6, the range of variations of the stresses σ_s on the contours of the circular holes increases from 4.5σ to 9.7σ for $\tau = 0.5\sigma$ (Figs. 3a, c). In the obtained optimal holes, the maximum relative deviations of the contour stresses σ_s from the mean stresses $\tilde{\sigma}_s$ ($\delta = \max |(\sigma_s(\xi_j) - \tilde{\sigma}_s)/\tilde{\sigma}_s| 100\%$) do not exceed 0.3, 0.7, and 1.5% for $\gamma = 1.4, 2.1$, and 2.6, respectively.

We determined the optimal shapes of holes in the quasiorthotropic plate ($\gamma = 2.1$) in the case of biaxial tension at infinity by mutually perpendicular stresses with different values: $p = \sigma$, $q = 0.75\sigma$ (Fig. 4a) and $p = \sigma$, $q = 0.5\sigma$ (Fig. 4b) for numerous angles of inclination α of the stresses p to the Ox -axis in the absence of shear loads at infinity ($\tau = 0$). In Fig. 5, we illustrate the corresponding distributions of the relative stresses σ_s/σ on the circular and optimal holes. In this case, for all analyzed angles α in the obtained optimal holes, the maximum relative deviations δ of the contour stresses from the mean stresses do not exceed 0.8% for the load $p = \sigma$ and $q = 0.75\sigma$ and 2.5% for $p = \sigma$ and $q = 0.5\sigma$.

The influence of the orthotropy parameter γ of the material of the plate subjected to biaxial tension and shear at infinity on the shapes of the optimal holes and the distributions of stresses along the contours of the circular

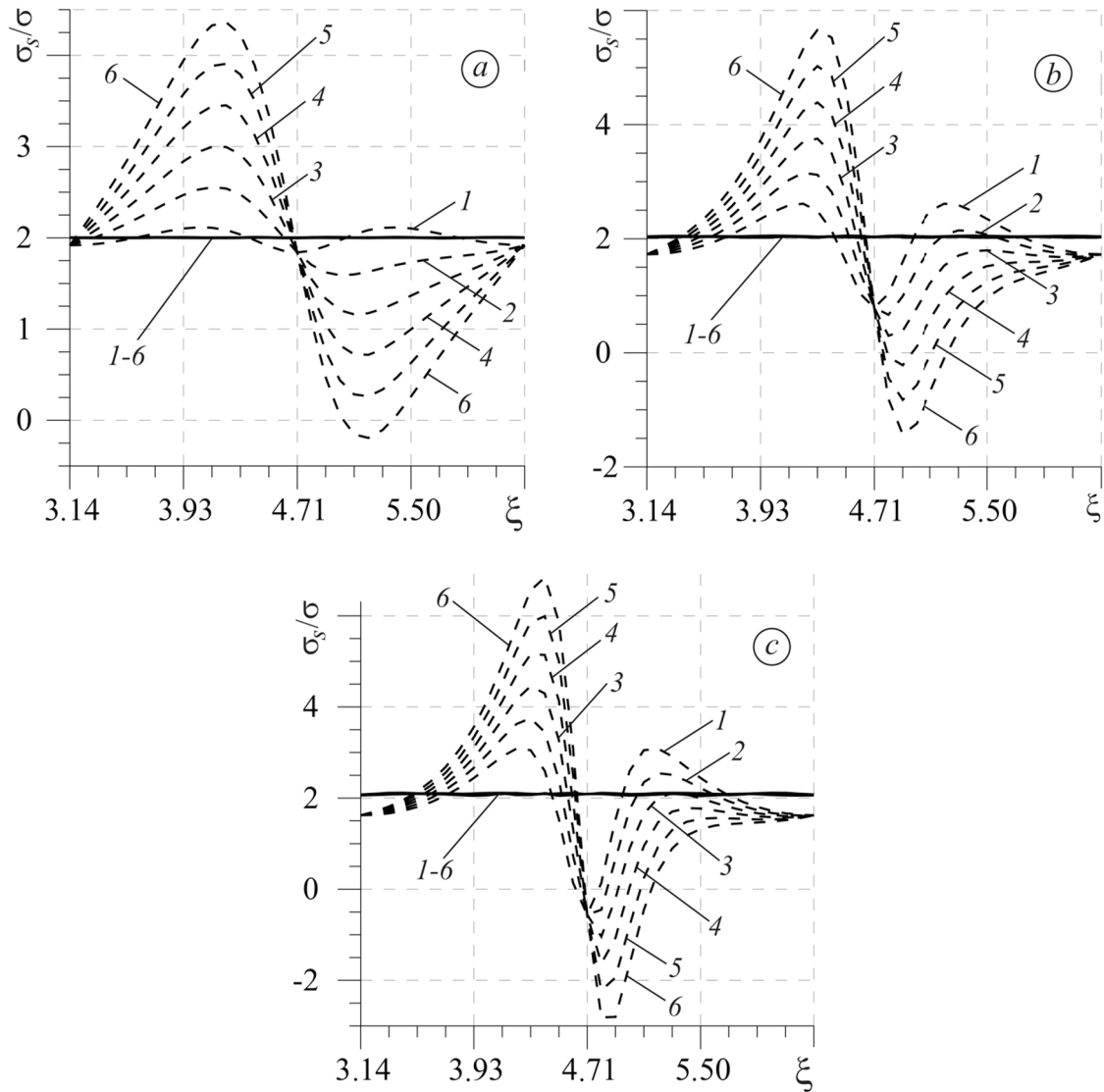


Fig. 3. Influence of the additional shear load ($\tau = \{0; 0.1; 0.2; 0.3; 0.4; 0.5\}\sigma$ correspond to curves 1–6) on the distributions of stresses σ_s/σ along the contours of circular (dashed lines) and optimal (solid lines) holes in the quasiorthotropic plate in the case of uniform tension ($p = q = \sigma$) for $\gamma = 1.4$ (a); 2.1 (b), and 2.6 (c).

and optimal holes is illustrated in Fig. 6. It is shown that this influence strongly affects the distributions of contour stresses over the edge of the given circular hole (Fig. 6b). As the orthotropy of the material of the plate increases, the range of stresses σ_s becomes much larger: increases from 2.2σ for the isotropic material ($\gamma = 1$) to 9.3σ for the quasiisotropic material ($\gamma = 3$). The influence of the parameter γ on the shapes of the optimal holes is much weaker (Fig. 6a). However, even for relatively small changes in the shape of the hole, the distribution of contour stresses σ_s undergoes significant changes. In this case, for different values of γ , the optimal holes are sought by using expansions (10) with respect to the parameters P_m ($m = 1, \dots, 10$). Their values for the optimal shapes of the holes and the maximum relative deviations δ of the contour stresses σ_s from the mean stresses $\bar{\sigma}_s$ are presented in Table 1.

The additional shear loading of the quasiorthotropic plate at infinity under the conditions of biaxial tension along the axes of orthotropy (see Figs. 2 and 6a), as in the case of biaxial tension at a certain angle to the axes of

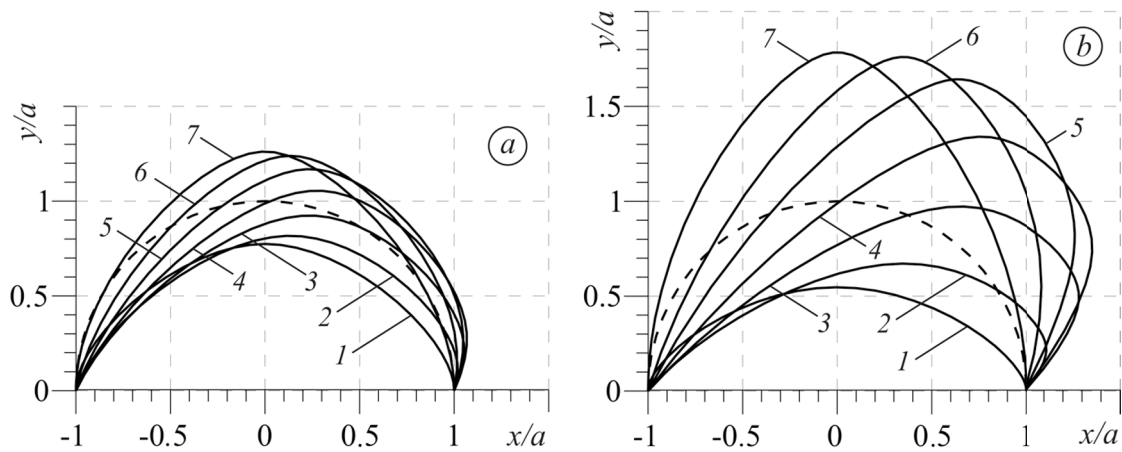


Fig. 4. Shapes of the optimal holes (their upper halves) and their comparison with a semicircle (dashed line) for the quasiorthotropic plate ($\gamma = 2.1$) under the conditions of biaxial tension with $p = \sigma$, $q = 0.75\sigma$ (a) or $p = \sigma$, $q = 0.5\sigma$ (b) for $\tau = 0$ and the following angles: $\alpha = 0; \pi/12; \pi/6; \pi/4; \pi/3; 5\pi/12; \pi/2$ (curves 1–7).

Table 1
Parameters of the Optimal Holes P_m ($m = 1, \dots, 10$) and the Maximum Relative Deviations δ of the Contour Stresses from the Mean Stresses under the Loads $p = \sigma$, $q = 0.75\sigma$ ($\alpha = \pi/2$), and $\tau = 0.25\sigma$

γ	P_1/P_6	P_2/P_7	P_3/P_8	P_4/P_9	P_5/P_{10}	δ (%)
1.0	1.002/1.399	-0.063/-0.226	0.010/0.213	-0.238/0.211	-0.008/-0.012	0.04
1.5	0.987/1.540	-0.703/0.019	0.027/-0.019	0.015/-0.027	-0.002/-0.012	0.29
2.0	0.857/1.615	-0.889/0.121	0.127/0.102	-0.080/-0.093	0.016/0.009	0.37
2.5	0.838/1.597	-0.924/0.153	0.139/0.103	-0.132/-0.113	0.021/0.017	0.85
3.0	0.798/1.628	-0.880/0.141	0.186/0.111	-0.206/-0.082	0.018/0.011	4.05

orthotropy (see Fig. 4), strongly affects the shapes of the optimal holes. In the absence of shear loads ($\tau = 0$) and biaxial tension of the quasiorthotropic plate along the axes of orthotropy by the stresses p and q ($\alpha = \pi/2$), we arrive at the results obtained in [9], whereas for the isotropic material ($\gamma = 1$) under biaxial tension and shear, we get the well-known optimal elliptic holes [1, 14]. In view of the qualitatively different distributions of the contour stresses σ_s in quasiorthotropic [9] and orthotropic [8, 16] plates subjected to the same biaxial tension, the shapes of the optimal holes in these plates are also different [8, 9].

CONCLUSIONS

We extended the earlier developed methods aimed at finding the optimal shapes of the holes in quasiorthotropic plates subjected to biaxial tension symmetric about the axes of orthotropy to the general case of tension of the plate in mutually perpendicular directions (by stresses with different values and directions of action relative

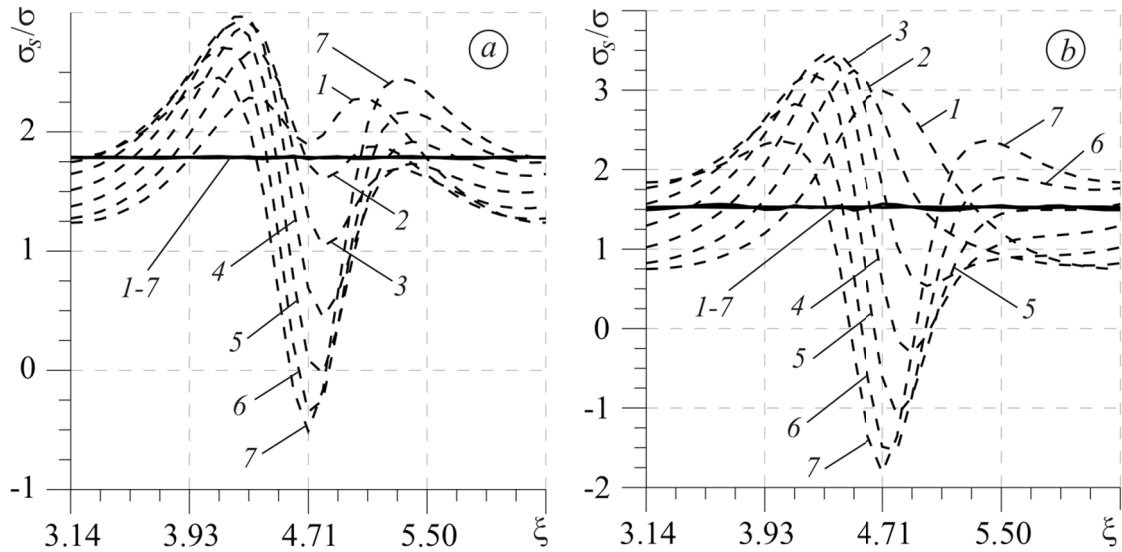


Fig. 5. Distributions of stresses σ_s/σ along the contours of circular (dashed lines) and optimal (solid lines) holes in the quasiorthotropic plate ($\gamma = 2.1$) under the conditions of biaxial tension (see Fig. 4).

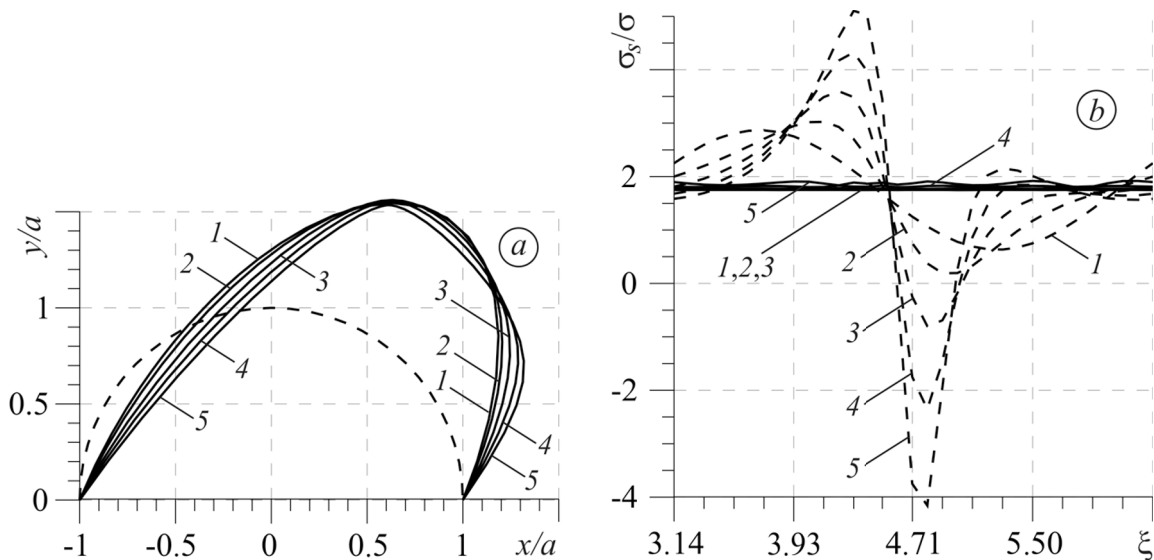


Fig. 6. The shapes of the optimal holes (upper halves) compared with the semicircle (dashed line) (a) and the distributions of the stresses σ_s/σ (b) over the contours of circular (dashed lines) and optimal (solid lines) holes in the case of biaxial tension $p = \sigma$, $q = 0.75\sigma$, $\alpha = \pi/2$ and shear $\tau = 0.25\sigma$ for different orthotropy parameters $\gamma = 1; 1.5; 2; 2.5; 3$ (curves 1–5).

to the axes of orthotropy) and transverse shear at infinity. The solution of the inverse problems of the theory of elasticity for quasiorthotropic plates with holes of unknown shape is reduced to the minimization of the multiparameter functional of standard deviations of the contour stresses from their mean values. For this purpose, we use the effective numerical solutions (by the method of singular integral equations) of the corresponding direct problems for plane quasiisotropic bodies with smooth holes. We determined almost optimal shapes of the holes in quasiorthotropic plates with different levels of orthotropy subjected to the simultaneous action of shear loads at infinity along the axes of orthotropy of the plate and biaxial tension directed at a certain angle to these axes.

Insignificant maximum relative deviations of the contour stresses from the mean stresses for the discovered holes confirm the optimality of their shapes. The developed methods and the obtained results are suitable for the design of structural elements with optimal (characterized by minimum stress concentrations) holes in contemporary mechanical engineering produced from different anisotropic (composite) materials.

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