

THERMOELASTIC STATE OF A THERMOSENSITIVE HOLLOW SPHERE UNDER THE CONDITIONS OF CONVECTIVE-RADIANT HEAT EXCHANGE WITH THE ENVIRONMENT

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The solution of the nonstationary problem of heat conduction is constructed for a thermosensitive hollow sphere in the process of convective-radiant heat exchange with media kept at constant temperatures. In this case, it is assumed that the heat-transfer and emissivity coefficients of the surfaces are functions of temperature. We study the influence of the thermal sensitivity of the material of the sphere on the intensity and character of distribution of temperature and the components of the stress-strain state.

Structural elements of contemporary equipment operate, as a rule, under the action of combined thermal and force loads. This is why the development of mathematical models aimed at the investigation of temperature stresses (in numerous cases, these stresses determine the serviceability of structures) is an urgent problem for the engineering practice. The classical model based on the assumption of invariability of the properties of materials inadequately describes the behavior of the bodies in the presence of significant space and time temperature gradients. The analysis of the thermal sensitivity of materials substantially complicates the problem because, in this case, we arrive at the necessity of solution of nonlinear boundary-value problems of mathematical physics and boundary-value problems for systems of differential equations with variable coefficients. The analysis of the literature sources (a brief survey can be found in [1]) shows that the existing approaches to the solution of problems of this sort are, as a rule, based on certain assumptions and restrictions imposed on the temperature dependences of characteristics of the material. This is why it is necessary to develop the methods taking into account the thermal sensitivity of all characteristics of the material and convenient for practical applications. Thus, in [1–3], we determine the temperature field and the stress–strain state caused by this field for a thermosensitive solid sphere, a layer, and a space containing a cavity under the conditions of convective-radiant heat exchange with the environment. The thermoelastic state of a thermosensitive hollow sphere in the process of convective heat exchange with the environment is numerically investigated in [4]. In the cited work, the solution of the problem of heat conduction is constructed by using the numerical-analytic procedure [5] for a thermosensitive hollow sphere under the conditions of complex heat exchange with media kept at different temperatures, and the thermoelastic state of the sphere is determined by the perturbation method. The influence of the temperature dependences of characteristics of the material of the sphere on the levels and character of distribution of temperature and temperature-induced stresses are studied both for a sphere free of external loads and in the presence of force loads applied to its surfaces.

Statement of the Problem

Consider the problem of determination of the nonstationary temperature field t and the stress–strain state

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induced by this field in a hollow sphere with inner radius r_1 and outer radius r_2 in the case where the thermo-mechanical characteristics of the material of the sphere are functions of temperature. The temperature t_p is uniformly distributed inside the sphere and its surfaces $r = r_1$ and $r = r_2$ are subjected to the action of constant pressures p_1 and p_2 , respectively. Beginning with time $\tau = 0$, the sphere exchanges heat (as a result of convective-radiant heat exchange through the surfaces $r = r_1$ and $r = r_2$) with media kept at constant temperatures t_{c1} and t_{c2} .

Determination of the Temperature Field

The indicated temperature field in the hollow sphere is determined from the nonlinear equation of heat conduction [6, 7]:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \lambda_t(t) \frac{\partial t}{\partial r} \right) = c_v(t) \frac{\partial t}{\partial \tau} \quad (1)$$

with the following boundary and initial conditions:

$$\left[\lambda_t(t) \frac{\partial t}{\partial r} + (-1)^j (\alpha_j(t)(t - t_{cj}) + \sigma \varepsilon_j(t)(t^4 - t_{cj}^4)) \right]_{r=r_j} = 0 \quad (j = 1, 2), \quad (2)$$

$$t|_{\tau=0} = t_p, \quad (3)$$

where $c_v(t)$ and $\lambda_t(t)$ are, respectively, the volumetric heat capacity and heat-conduction coefficient of the material of the sphere regarded as functions of temperature, $\alpha_j(t)$ are the heat-transfer coefficients through the surfaces $r = r_j$ ($j = 1, 2$) regarded as functions of temperature, $\varepsilon_j(t)$ are the emissivity coefficients of the corresponding surfaces also regarded as functions of temperature, and σ is the Stefan–Boltzmann constant.

As a reference temperature, we choose its value t_0 . As the characteristic size of the sphere, we take the thickness of its wall $l_0 = r_2 - r_1$. Further, we introduce the dimensionless temperature $T = t/t_0$ and dimensionless coordinate $\rho = r/l_0$ and represent the characteristics of the material of the sphere, the heat-transfer coefficients, and the emissivity coefficients of the surfaces in the form $\chi(t) = \chi_0 \chi^*(t)$, where the quantities with subscript “0” have the required dimensionality and the quantities with asterisks are functions of dimensionless temperature. In this case, $\chi(t_p) = \chi_0$ and, hence, $\chi^*(t_p) = 1$ ($T_p = t_p/t_0$). As a result, problem (1)–(3) takes the form

$$\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \lambda_t^*(T) \frac{\partial T}{\partial \rho} \right) = c_v^*(T) \frac{\partial T}{\partial \text{Fo}}, \quad (4)$$

$$\left[\lambda_t^*(T) \frac{\partial T}{\partial \rho} + (-1)^j (\text{Bi}_j \alpha_j^*(T)(T - T_{cj}) + \text{Sk}_j \varepsilon_j^*(T)(T^4 - T_{cj}^4)) \right]_{\rho=\rho_j} = 0 \quad (j = 1, 2), \quad (5)$$

$$T|_{\text{Fo}=0} = T_p, \quad (6)$$

where

$$T_{cj} = \frac{t_{cj}}{t_0}, \quad \text{Bi}_j = \frac{\alpha_{0j} l_0}{\lambda_{t0}}, \quad \text{Sk}_j = \frac{\sigma \varepsilon_{0j} l_0^3}{\lambda_{t0}}, \quad \text{Fo} = \frac{a_0 \tau}{l_0^2}, \quad \text{and} \quad a_0 = \frac{\lambda_{t0}}{c_{v0}}.$$

Let us now apply the Kirchhoff transformation to the nonlinear problem (4)–(6):

$$\theta = \int_{T_p}^T \lambda_t^*(T) dt. \quad (7)$$

As a result, we arrive at the following boundary value-problem with respect to the variable θ :

$$\frac{\partial^2(\rho\theta)}{\partial \rho^2} = \frac{1}{a^*(\theta)} \frac{\partial(\rho\theta)}{\partial \text{Fo}}, \quad (8)$$

$$\left[\frac{\partial \theta}{\partial \rho} + (-1)^j (\text{Bi}_j \alpha_j^*(T(\theta))(T(\theta) - T_{cj}) + \text{Sk}_j \varepsilon_j^*(T(\theta))(T^4(\theta) - T_{cj}^4)) \right]_{\rho=\rho_j} = 0 \quad (j = 1, 2), \quad (9)$$

$$\theta|_{\text{Fo}=0} = 0, \quad (10)$$

where $T(\theta)$ denotes the dependence of the dimensionless temperature on the Kirchhoff variable θ [for the specific dependence of the heat-conduction coefficient on temperature, the function $T(\theta)$ can be found from the integral equation (7)] and

$$a^*(t) = \frac{\lambda_t^*[T(\theta)]}{c_v^*[T(\theta)]}.$$

To find the solution of the boundary-value problem with respect to the variable θ , we use the method of successive approximations. As the m th ($m = 1, 2, \dots$) approximation to the solution of the problem, we use the analytic solution of the following linear problem:

$$\frac{\partial^2(\rho\theta_m)}{\partial \rho^2} = \frac{\partial(\rho\theta_m)}{\partial \text{Fo}_m}, \quad (11)$$

$$\left[\frac{\partial \theta_m}{\partial \rho} + (-1)^j \text{Bi}_{jm} (\theta_m - \theta_{cj}) \right]_{\rho=\rho_j} = 0 \quad (j = 1, 2), \quad (12)$$

$$\theta_m|_{\text{Fo}_m=0} = 0, \quad (13)$$

where

$$\theta_{cj} = \int_{T_p}^{T_{cj}} \lambda_t^*(T) dt, \quad \text{Bi}_{1j} = \text{Bi}_j, \quad \text{Fo}_1 = \text{Fo},$$

$$\begin{aligned} \text{Bi}_{jm} = & [\theta_{m-1}(\text{Fo}^*, \rho_j) - \theta_{cj}]^{-1} \left\{ \text{Bi}_j \alpha_j^*(T(\theta_{m-1}(\text{Fo}^*, \rho_j))) [T(\theta_{m-1}(\text{Fo}^*, \rho_j)) - T_{cj}] \right. \\ & \left. + \text{Sk}_j \varepsilon_j^*(T(\theta_{m-1}(\text{Fo}^*, \rho_j))) [T^4(\theta_{m-1}(\text{Fo}^*, \rho_j)) - T_{cj}^4] \right\}, \quad j = 1, 2, \\ \text{Fo}_m = & a^*(\theta_{m-1}(\text{Fo}^*, \rho_j)) \text{Fo} \quad (m \geq 2), \end{aligned}$$

and Fo^* is the time for which we determine the values of temperature.

To find the solution of problem (11)–(13), we use the integral Laplace transformation with respect to the variable Fo_m . As a result, we arrive at the following boundary-value problem for the transform of the Kirchhoff variable:

$$\frac{d^2(\rho \tilde{\theta}_m)}{d\rho^2} - s\rho \tilde{\theta}_m = 0, \quad (14)$$

$$\left[\frac{d\tilde{\theta}_m}{d\rho} + (-1)^j \text{Bi}_{jm} \left(\tilde{\theta}_m - \frac{\theta_{cj}}{s} \right) \right]_{\rho=\rho_j} = 0, \quad (15)$$

where

$$\tilde{\theta}_m = \int_0^\infty \theta_m e^{-s\text{Fo}_m} d\text{Fo}_m$$

and s is the parameter of the Laplace transformation.

The solution of Eq. (14) is sought in the form

$$\tilde{\theta}_m = \frac{A \cosh \sqrt{s\rho} + B \sinh \sqrt{s\rho}}{\rho}.$$

The constants of integration A and B are found from the boundary conditions (15) and, as a result, we get

$$\tilde{\theta}_m = \frac{\phi(s)}{\rho s \varphi(s)}, \quad (16)$$

where

$$\begin{aligned} \phi(s) = & \text{Bi}_{1m} \rho_1^2 \theta_{c1} m_2(s) + \text{Bi}_{2m} \rho_2^2 \theta_{c2} m_1(s), \\ m_j(s) = & \frac{(1 - (-1)^j \rho_j \text{Bi}_{jm}) \sinh \sqrt{s}(\rho - \rho_j)}{\sqrt{s}} + \rho_j \cosh \sqrt{s}(\rho - \rho_j) \quad (j = 1, 2), \\ \varphi(s) = & \frac{[\rho_1 \rho_2 s - (1 + \rho_1 \text{Bi}_{1m})(1 - \rho_2 \text{Bi}_{2m})] \sinh \sqrt{s}(\rho_2 - \rho_1)}{\sqrt{s}} \\ & + [\rho_2(1 + \rho_1 \text{Bi}_{1m}) - \rho_1(1 - \rho_2 \text{Bi}_{2m})] \cosh \sqrt{s}(\rho_2 - \rho_1). \end{aligned}$$

Since solution (16) is the ratio of generalized polynomials in the parameter s and the polynomial in the denominator does not contain the free term, the inverse Laplace transformation can be performed by using the Vashchenko–Zakharchenko theorem on expansion [8]. The denominator possesses a simple root $s = 0$ and infinitely many simple roots $s_n = -\mu_{nm}^2$ ($\mu_{nm} = i\sqrt{s}$), where μ_{nm} are the roots of the characteristic equation

$$\cot(\rho_2 - \rho_1)\mu = \frac{(1 + \rho_1 \text{Bi}_{1m})(1 - \rho_2 \text{Bi}_{2m}) + \rho_1 \rho_2 \mu^2}{[\rho_2(1 + \rho_1 \text{Bi}_{1m}) - \rho_1(1 - \rho_2 \text{Bi}_{2m})]\mu}$$

obtained from the equation $\varphi(s) = 0$.

This enables us to conclude that

$$\theta_m = \frac{1}{\rho} \left[\frac{\phi(0)}{\varphi(0)} - \sum_{n=1}^{\infty} \frac{\phi(\mu_{nm})}{\mu_{nm}^2 \varphi'(\mu_{nm}) e^{\mu_{nm}^2 \text{Fo}_m}} \right], \quad (17)$$

where

$$\phi(\gamma) = \text{Bi}_{1m} \rho_1^2 \theta_{c1} m_2(\gamma) + \text{Bi}_{2m} \rho_2^2 \theta_{c2} m_1(\gamma) \quad (\gamma = 0, \mu_{nm}),$$

$$m_j(0) = (1 - (-1)^j \rho_j \text{Bi}_{jm})(\rho - \rho_j) + \rho_j,$$

$$m_j(\mu_{nm}) = (1 - (-1)^j \rho_j \text{Bi}_{jm}) \mu_{nm}^{-1} \sin \mu_{nm} (\rho - \rho_j) + \rho_j \cos \mu_{nm} (\rho - \rho_j) \quad (j = 1, 2),$$

$$\varphi(0) = \rho_2(1 + \rho_1 \text{Bi}_{1m}) - \rho_1(1 - \rho_2 \text{Bi}_{2m}) + (1 + \rho_1 \text{Bi}_{1m})(1 - \rho_2 \text{Bi}_{2m})(\rho_2 - \rho_1),$$

$$\begin{aligned} \varphi'(\mu_{nm}) = & 0.5 \{ [\rho_1 \rho_2 + (\rho_2(1 + \rho_1 \text{Bi}_{1m}) - \rho_1(1 - \rho_2 \text{Bi}_{2m}))(\rho_2 - \rho_1) \\ & - (1 + \rho_1 \text{Bi}_{1m})(1 - \rho_2 \text{Bi}_{2m}) \mu_{nm}^{-2}] \mu_{nm}^{-1} \sin \mu_{nm} (\rho_2 - \rho_1) \\ & + (\rho_1 \rho_2 + (1 + \rho_1 \text{Bi}_{1m})(1 - \rho_2 \text{Bi}_{2m}) \mu_{nm}^{-2})(\rho_2 - \rho_1) \cos \mu_{nm} (\rho_2 - \rho_1) \}. \end{aligned}$$

If, e.g., the heat-conduction coefficient is a linear function of temperature, i.e., $\lambda_t^*(T) = 1 + k(T - T_p)$ ($k = \text{const}$), then the m th approximation of temperature in the sphere is given by the formula

$$T_m = \frac{\sqrt{1 + 2k\theta_m} - 1}{k} + T_p. \quad (18)$$

In the case where a constant temperature t_{c1} is set on the surface of the cavity $r = r_1$ and the process of convective-radiant heat exchange with the environment kept at a constant temperature t_{c2} runs through the surface of the sphere $r = r_2$, the Kirchhoff variable has the form specified by relation (17) with

$$\phi(0) = \text{Bi}_{2m} \rho_2^2 \theta_{c2} (\rho - \rho_1) + \theta_{c1} \rho_1 [\rho_2 + (1 - \text{Bi}_{2m} \rho_2)(\rho - \rho_2)],$$

$$\varphi(0) = \rho_2 - (1 - \rho_2 \text{Bi}_{2m})(\rho_2 - \rho_1),$$

$$\phi(\mu_{nm}) = \text{Bi}_{2m} \rho_2^2 \theta_{c2} \mu_{nm}^{-1} \sin \mu_{nm}(\rho - \rho_1) + \theta_{c1} \rho_1 [\rho_2 \cos \mu_{nm}(\rho - \rho_2) + (1 - \text{Bi}_{2m} \rho_2) \mu_{nm}^{-1} \sin \mu_{nm}(\rho - \rho_2)],$$

$$\begin{aligned} \varphi'(\mu_{nm}) = 0.5 \{ & [\mu_{nm}^{-1} \sin \mu_{nm}(\rho_2 - \rho_1) [\rho_2(\rho_2 - \rho_1) - \mu_{nm}^{-2} (1 - \text{Bi}_{2m} \rho_2)] \\ & + \mu_{nm}^{-2} \cos \mu_{nm}(\rho_2 - \rho_1) (1 - \text{Bi}_{2m} \rho_2)(\rho_2 - \rho_1) \}, \end{aligned}$$

and μ_{nm} are the roots of the characteristic equation $\mu \cot \mu(\rho_2 - \rho_1) = (1 - \text{Bi}_{2m} \rho_2) \rho_2^{-1}$.

If, on the contrary, the outer surface of the sphere $r = r_2$ is kept at a constant temperature t_{c2} and the convective-radiant heat exchange with the environment kept at temperature t_{c1} is realized through the surface $r = r_1$, then the Kirchhoff variable is given by relation (17) with

$$\phi(0) = -\text{Bi}_{1m} \rho_1^2 \theta_{c1}(\rho - \rho_2) + \theta_{c2} \rho_2 [\rho_1 + (1 + \text{Bi}_{1m} \rho_1)(\rho - \rho_1)],$$

$$\varphi(0) = \rho_1 + (1 + \rho_1 \text{Bi}_{1m})(\rho_2 - \rho_1),$$

(19)

$$\phi(\mu_{nm}) = -\text{Bi}_{1m} \rho_1^2 \theta_{c1} \mu_{nm}^{-1} \sin \mu_{nm}(\rho - \rho_2)$$

$$+ \theta_{c2} \rho_2 [\rho_1 \cos \mu_{nm}(\rho - \rho_2) + (1 + \text{Bi}_{1m} \rho_1) \mu_{nm}^{-1} \sin \mu_{nm}(\rho - \rho_1)],$$

$$\begin{aligned} \varphi'(\mu_{nm}) = 0.5 \{ & [\mu_{nm}^{-1} \sin \mu_{nm}(\rho_2 - \rho_1) [\rho_1(\rho_2 - \rho_1) + \mu_{nm}^{-2} (1 + \text{Bi}_{1m} \rho_1)] \\ & - \mu_{nm}^{-2} \cos \mu_{nm}(\rho_2 - \rho_1) (1 + \text{Bi}_{1m} \rho_1)(\rho_2 - \rho_1) \}, \end{aligned}$$

and μ_{nm} are the roots of the characteristic equation $\mu \cot \mu(\rho_2 - \rho_1) = (1 + \text{Bi}_{1m} \rho_1) \rho_1^{-1}$.

Numerical Solution of the Problem of Heat Conduction

For the sake of comparison, we also find the numerical solution of the problem for the Kirchhoff variable (8)–(10). To construct a semidiscrete model of this problem, we use the integro-interpolation method.

The role of difference analog of Eq. (8) in a segment $[\rho_1, \rho_2]$ of the domain of variation of the variable ρ in a grid

$$w_h = \left\{ \rho_i = \rho_1 + ih, \quad i = 0, 1, \dots, n, \quad h = \frac{\rho_2 - \rho_1}{n} \right\}$$

is played by the following system of ordinary differential equations [3]:

$$\frac{d\theta_i}{d\text{Fo}} = a^*(\theta_i) \frac{1}{h^2(1 + ih)^2} [(\theta_{i+1} - \theta_i) \rho_{i+1/2}^2 - (\theta_i - \theta_{i-1}) \rho_{i-1/2}^2] \quad (20)$$

We now construct a difference analog of the boundary condition (9) for $\rho = \rho_1$. To do this, we integrate Eq. (8) with respect to the variable ρ over the segment $\rho_0 \leq \rho \leq \rho_{1/2}$ ($\rho_0 = \rho_1$, $\rho_{1/2} = \rho_1 + 0.5h$). As a result, we find

$$\int_{\rho_0}^{\rho_{1/2}} \frac{1}{a^*(\theta)} \rho^2 \frac{\partial \theta}{\partial \text{Fo}} d\rho = v_{1/2} - v_0, \quad (21)$$

where

$$v_0 \approx \rho_0^2 [\text{Bi}_1 \alpha_1^*(T(\theta_0))(T(\theta_0) - T_{c1}) + \text{Sk}_1 \varepsilon_1^*(T(\theta_0))(T^4(\theta_0) - T_{c1}^4)], \quad v_{1/2} \approx \frac{\theta_1 - \theta_0}{h} \rho_{1/2}^2, \quad (22)$$

with $T(\theta_0) = T(\theta)|_{\rho=\rho_0}$, $\theta_1 = \theta|_{\rho=\rho_1}$, and $\theta_0 = \theta|_{\rho=\rho_0}$. We set

$$\int_{\rho_0}^{\rho_{1/2}} \frac{1}{a^*(\theta)} \rho^2 \frac{\partial \theta}{\partial \text{Fo}} d\rho \approx \frac{h \rho_0^2}{2a^*(\theta_0)} \frac{d\theta_0}{d\text{Fo}}. \quad (23)$$

Equating the right-hand sides of Eqs. (21) and (23) and taking into account relation (22), we arrive at the difference analog of the boundary conditions (9) for $\rho = \rho_1$ in the following form:

$$\frac{d\theta_0}{d\text{Fo}} = a^*(\theta_0) \frac{2}{h} \left[\frac{1}{h \rho_0^2} (\theta_1 - \theta_0) \rho_{1/2}^2 - \text{Bi}_1 \alpha_1^*(T(\theta_0))(T(\theta_0) - T_{c1}) - \text{Sk}_1 \varepsilon_1^*(T(\theta_0))(T^4(\theta_0) - T_{c1}^4) \right].$$

For $\rho = \rho_2$, the difference analog of the boundary condition (9) is constructed in a similar way. For this purpose, we integrate Eq. (8) with respect to the variable ρ over the segment $\rho_{n-1/2} \leq \rho \leq \rho_n$. This gives

$$\int_{\rho_{n-1/2}}^{\rho_n} \frac{1}{a^*(\theta)} \rho^2 \frac{\partial \theta}{\partial \text{Fo}} d\rho \approx v_n - v_{n-1/2}, \quad (24)$$

where

$$v_n \approx \rho_n^2 [\text{Bi}_2 \alpha_2^*(T(\theta_n))(T(\theta_n) - T_{c2}) + \text{Sk}_2 \varepsilon_2^*(T(\theta_n))(T^4(\theta_n) - T_{c2}^4)] \quad \text{and} \quad v_{n-1/2} \approx \frac{\theta_n - \theta_{n-1}}{h} \rho_{n-1/2}^2 \quad (25)$$

with $T(\theta_n) = T(\theta)|_{\rho=\rho_n}$, $\theta_n = \theta|_{\rho=\rho_n}$, and $\theta_{n-1} = \theta|_{\rho=\rho_{n-1}}$. By setting

$$\int_{\rho_{n-1/2}}^{\rho_n} \frac{1}{a^*(\theta)} \rho^2 \frac{\partial \theta}{\partial \text{Fo}} d\rho \approx \frac{h \rho_n^2}{2a^*(\theta_n)} \frac{d\theta_n}{d\text{Fo}} \quad (26)$$

and equating the right-hand sides of Eqs. (24) and (26), in view of relation (25), we arrive at the following difference analog of the boundary condition (9) for $\rho = \rho_2$:

$$\frac{d\theta_n}{dFo} = -a^*(\theta_0) \frac{2}{h} \left[\text{Bi}_2 \alpha_2^*(T(\theta_n))(T(\theta_n) - T_{c2}) + \text{Sk}_2 \varepsilon_2^*(T(\theta_n))(T^4(\theta_n) - T_{c2}^4) + \frac{1}{h\rho_n^2}(\theta_n - \theta_{n-1})\rho_{n-1/2}^2 \right].$$

Thus, the semidiscrete model of the problem of finding the Kirchhoff variable for the thermosensitive hollow sphere constructed by the integro-interpolation method has the form of the following Cauchy problem for the system of ordinary differential equations:

$$\left\{ \begin{array}{l} \frac{d\theta_0}{dFo} = \frac{2a^*(\theta_0)}{h} \left[\frac{1}{h\rho_0^2}(\theta_1 - \theta_0)\rho_{1/2}^2 - \text{Bi}_1 \alpha_1^*(T(\theta_0))(T(\theta_0) - T_{c1}) - \text{Sk}_1 \varepsilon_1^*(T(\theta_0))(T^4(\theta_0) - T_{c1}^4) \right], \\ \frac{d\theta_i}{dFo} = \frac{a^*(\theta_i)}{h^2(\rho_1 + ih)^2} [(\theta_{i+1} - \theta_i)\rho_{i+1/2}^2 - (\theta_i - \theta_{i-1})\rho_{i-1/2}^2], \quad i = 1, \dots, n-1, \\ \frac{d\theta_n}{dFo} = -\frac{2a^*(\theta_0)}{h} \left[\text{Bi}_2 \alpha_2^*(T(\theta_n))(T(\theta_n) - T_{c2}) + \text{Sk}_2 \varepsilon_2^*(T(\theta_n))(T^4(\theta_n) - T_{c2}^4) + \frac{\theta_n - \theta_{n-1}}{h\rho_n^2} \rho_{n-1/2}^2 \right], \\ \theta_i(0) = 0. \end{array} \right. \quad (27)$$

Problem (27) is a stiff system of ordinary differential equations [9, 10]. At present, there exist powerful methods aimed at the solution of systems of this sort including, in particular, the Adams and Geer methods realized in programs capable of monitoring the accuracy of numerical solutions with automatic choice of the step and order of the method required to solve the problem with minimum losses of computer time.

Stress–Strain State of the Hollow Sphere

Assume that the mechanical characteristics of the sphere (its coefficient of linear expansion $\alpha_r(t)$, shear modulus $G(t)$, and Poisson's ratio $\nu(t)$) are functions of temperature and that the sphere suffers the action of both the already determined centrally symmetric temperature field and constant pressures p_1 and p_2 applied to the surfaces $\rho = \rho_1$ and $\rho = \rho_2$, respectively. In view of the symmetry of the problem, its stress-strain state in the spherical coordinate system is determined by the radial displacements $u(r, \tau)$, radial stresses σ_{rr} , and hoop stresses $\sigma_{\varphi\varphi}$ [7, 11].

We represent the mechanical characteristics in the form $\chi(t) = \chi_0 \chi^*(t)$ and introduce the following dimensionless quantities:

$$\bar{u} = \frac{u}{l_0 \alpha_0 t_0}, \quad \sigma_p = \frac{\sigma_{rr}}{2G_0 \alpha_0 t_0}, \quad \sigma_\phi = \frac{\sigma_{\varphi\varphi}}{2G_0 \alpha_0 t_0}, \quad \text{and} \quad \Phi^*(t) = \frac{1 + \nu(T)}{1 - \nu(T)} \int_{T_p}^T a^*(T) dT.$$

As a result, we get the following formulas for the temperature stresses [12]:

$$\sigma_p = \bar{G}(T) \left[(1 - \nu(T)) \frac{\partial \bar{u}}{\partial \rho} + \frac{2\nu(T) \bar{u}}{\rho} - (1 - \nu(T)) \Phi^*(T) \right],$$

$$\sigma_\phi = \bar{G}(T) \left[v(T) \frac{\partial \bar{u}}{\partial \rho} + \frac{\bar{u}}{\rho} - (1 - v(T)) \Phi^*(T) \right],$$

where

$$\bar{G}(T) = \frac{G^*(T)}{1 - 2v(T)}$$

and the displacements $\bar{u}$ are found from the equation

$$\frac{\partial}{\partial \rho} \left(\frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 \bar{u}) \right) = \frac{\partial \Phi^*}{\partial \rho} - \psi(T) \left(\frac{\partial \bar{u}}{\partial \rho} + 2m(T) \frac{\bar{u}}{\rho} - \Phi^*(T) \right), \quad (28)$$

where, in turn,

$$\psi(T) = \frac{\partial}{\partial \rho} \left(\ln \left(G^*(T) \frac{1 - v(T)}{1 - 2v(T)} \right) \right),$$

$$m(T) = \frac{\frac{\partial}{\partial \rho} \left(G^*(T) \frac{v(T)}{1 - 2v(T)} \right)}{\frac{\partial}{\partial \rho} \left(G^*(T) \frac{1 - v(T)}{1 - 2v(T)} \right)}.$$

The solution of Eq. (28) obtained by the perturbation method [3] has the form

$$\bar{u} = \sum_{k=0}^{\infty} u_k, \quad (29)$$

where

$$u_0 = c_{10} \rho + \frac{1}{\rho^2} \left(c_{20} + H(\rho) - \frac{H_3(\rho)}{3} \right) + \frac{\rho H_0(\rho)}{3},$$

$$u_k = c_{1k} \rho + \frac{1}{\rho^2} \left(c_{2k} + \frac{H_3^{k-1}(\rho)}{3} \right) - \frac{\rho H_0^{k-1}(\rho)}{3},$$

$$H(\rho) = \int_{\rho_1}^{\rho} \xi^2 \Phi^*(\xi, \text{Fo}) d\xi, \quad H_m(\rho) = \int_{\rho_1}^{\rho} \xi^m \psi(T) \Phi^*(\xi, \text{Fo}) d\xi,$$

$$H_m^{(k-1)}(\rho) = \int_{\rho_1}^{\rho} \xi^m f_{k-1}(\xi, \text{Fo}) d\xi.$$

Hence, for the temperature stresses, we find

$$\sigma_{\rho} = \sum_{k=0}^{\infty} \sigma_{\rho k} \quad \text{and} \quad \sigma_{\phi} = \sum_{k=0}^{\infty} \sigma_{\phi k}, \quad (30)$$

where

$$\begin{aligned} \sigma_{\rho 0} &= G^*(T) \left[\bar{v}(T) \left(c_{10} + \frac{H_0(\rho)}{3} \right) - \frac{2}{\rho^3} \left(c_{20} + H(\rho) - \frac{H_3(\rho)}{3} \right) \right], \\ \sigma_{\phi 0} &= G^*(T) \left[\bar{v}(T) \left(c_{10} + \frac{H_0(\rho)}{3} \right) + \frac{1}{\rho^3} \left(c_{20} + H(\rho) - \frac{H_3(\rho)}{3} \right) - \Phi^*(T) \right], \\ \sigma_{\rho k} &= G^*(T) \left[\bar{v}(T) \left(c_{1k} - \frac{H_0^{k-1}(\rho)}{3} \right) - \frac{2}{\rho^3} \left(c_{2k} + \frac{H_3^{k-1}(\rho)}{3} \right) \right], \\ \sigma_{\phi k} &= G^*(T) \left[\bar{v}(T) \left(c_{1k} - \frac{H_0^{k-1}(\rho)}{3} \right) + \frac{1}{\rho^3} \left(c_{2k} + \frac{H_3^{k-1}(\rho)}{3} \right) \right], \end{aligned}$$

and

$$\bar{v}(T) = \frac{1 + v(T)}{1 - 2v(T)}.$$

The constants of integration c_{ik} ($i = 1, 2$) are found from the conditions imposed on the inner and outer surfaces of the sphere:

$$\sigma_{\rho k} \big|_{\rho=\rho_i} = -\bar{p}_i \delta_{ik}, \quad \text{where} \quad \bar{p}_i = \frac{p_i}{2G_0 \alpha_{t0} t_0}$$

and δ_{ik} is the Kronecker symbol. As a result, we obtain

$$\begin{aligned} c_{10} &= \frac{B_1 \rho_1^3 - B_2 \rho_2^3}{\Delta}, \quad c_{1k} = -\frac{\rho_2^3 B_{k-1}}{\Delta}, \quad c_{20} = -\frac{\rho_1^3 \rho_2^3 (\bar{v}_1 B_2 - \bar{v}_2 B_1)}{2\Delta}, \\ c_{2k} &= -\frac{\bar{v}_1 \rho_1^3 \rho_2^3 B_{k-1}}{2\Delta}, \quad B_1 = -\frac{\bar{p}_1}{G_1}, \quad B_2 = \frac{2H(\rho_2)}{\rho_2^3} - \frac{\bar{p}_2}{G_2} - \frac{\bar{v}_2 H_0(\rho_2) + 2\rho_2^{-3} H_3(\rho_2)}{3}, \\ \Delta &= \bar{v}_1 \rho_1^3 - \bar{v}_2 \rho_2^3, \quad G_i = G^*(T) \big|_{\rho=\rho_i}, \quad B_{k-1} = \frac{\bar{v}_2 H_0^{k-1}(\rho_2) + 2H_3^{k-1}(\rho_2) \rho_2^{-3}}{3}, \quad \bar{v}_i = \bar{v}(T) \big|_{\rho=\rho_i}. \end{aligned}$$

Numerical Investigation

As a numerical example, we consider a thermosensitive hollow sphere in the state of convective-radiant heat exchange through the surface of the cavity with the environment kept at a constant temperature $t_{c1} = 873^\circ\text{K}$.

The temperature of the outer surface of the sphere is equal to the temperature of the environment $t_{\infty} = 373^{\circ}\text{K}$. The heat-transfer and emissivity coefficients are regarded as constant.

The sphere is made of U12 steel. The reference values of its thermal properties and mechanical characteristics for different temperatures are presented in [13, 14]. To represent these characteristics in the form $\chi(t) = \chi_0 \chi^*(t)$, it is necessary to approximate the experimental data by analytic expressions. For this purpose, we use the built-in functions of linear and polynomial regression of the Excel spreadsheet editor from the Microsoft-Office Package for Windows (Fig. 1). By using regression functions for the approximation of the data, the residual standard deviation of the predicted (by the least-squares method) values from the actual data is minimized. It is given by the formula

$$E = \sum_{i=1}^n (y(x_i) - y_i)^2,$$

where $y(x_i)$ are the predicted values, n is the number of points, and x_i and y_i are the actual values. By using Excel, we also automatically determine the additional regression statistics and the determinacy coefficient R^2 specifying the accuracy of description of the actual data by the equation obtained with the help of regression analysis. This coefficient takes values from 0 to 1. The value 1 corresponds to the complete coincidence of the predicted and actual data. The approximation is regarded as good whenever $R^2 > 0.9$. The determinacy coefficient is given by the formula

$$R^2 = 1 - \frac{\sum_{i=1}^n (y_i - y(x_i))^2}{\sum_{i=1}^n (y_i - \langle y \rangle)^2},$$

where

$$\langle y \rangle = \sum_{i=1}^n \frac{y_i}{n}.$$

Hence, if we assume that the initial temperature is equal to 373°K and the reference temperature to 873°K , then, by using the deduced analytic relations, we represent the expressions for the thermal and mechanical characteristics of U12 steel as follows:

$$\lambda_t(t) = 45.04 (1 - 0.51(T - T_p)) \text{ [W/(m} \cdot \text{K)]},$$

$$a(t) = 11.42 \cdot 10^{-6} (1 - 0.86(T - T_p)) \text{ [m}^2\text{/sec]},$$

$$\alpha_t(t) = 11.68 \cdot 10^{-6} (1 + 1.33(T - T_p) - 0.65(T - T_p)^2) \text{ [1/K]},$$

$$G(t) = 0.794 \cdot 10^{11} (1 - 0.27(T - T_p) + 0.21(T - T_p)^2 + 0.59(T - T_p)^3) \text{ [Pa]},$$

$$v(t) = 0.282 (1 + 0.199(T - T_p) - 1.291(T - T_p)^2 + 2.36(T - T_p)^3).$$

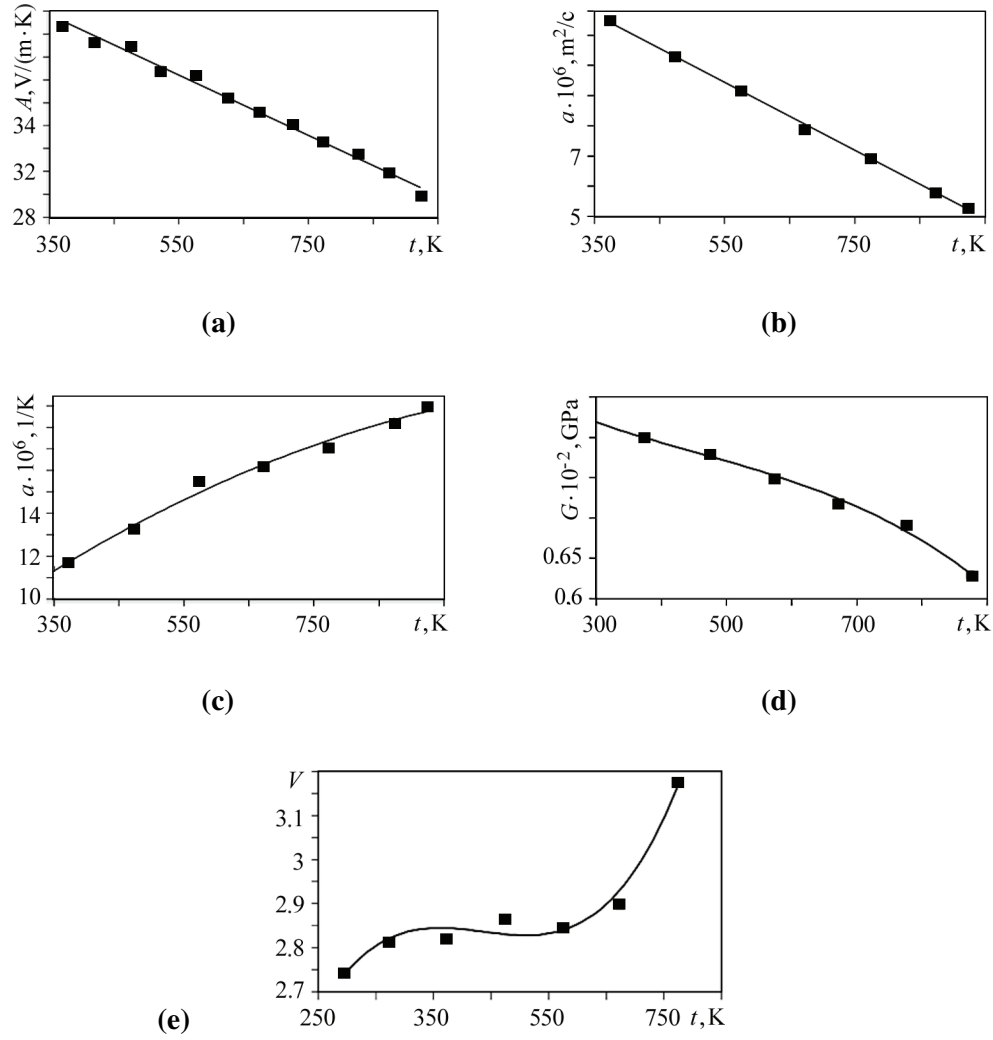


Fig. 1. Experimental data and approximating curves for the temperature dependences of the characteristics of U12 steel: (a) heat-conduction coefficient, (b) thermal diffusivity, (c) linear-expansion coefficient, (d) shear modulus, (e) Poisson's ratio.

Then the temperature field of the hollow sphere is found by using relation (18) with θ_m given by relations (17) and (19). The corresponding semidiscrete model for the numerical evaluation of the Kirchhoff variable is reduced to the following Cauchy problem for the system of ordinary differential equations:

$$\begin{cases} \frac{d\theta_0}{dFo} = \frac{2a^*(\theta_0)}{h} \left[\frac{1}{h\rho_0^2} (\theta_1 - \theta_0)\rho_{1/2}^2 - Bi_1(T(\theta_0) - T_{c1}) - Sk_1(T^4(\theta_0) - T_{c1}^4) \right], \\ \frac{d\theta_i}{dFo} = \frac{a^*(\theta_i)}{h^2(\rho_1 + ih)^2} [(\theta_{i+1} - \theta_i)\rho_{i+1/2}^2 - (\theta_i - \theta_{i-1})\rho_{i-1/2}^2], \quad i = 1, \dots, n-1, \\ \frac{d\theta_n}{dFo} = 0, \\ \theta_i(0) = 0. \end{cases} \quad (31)$$

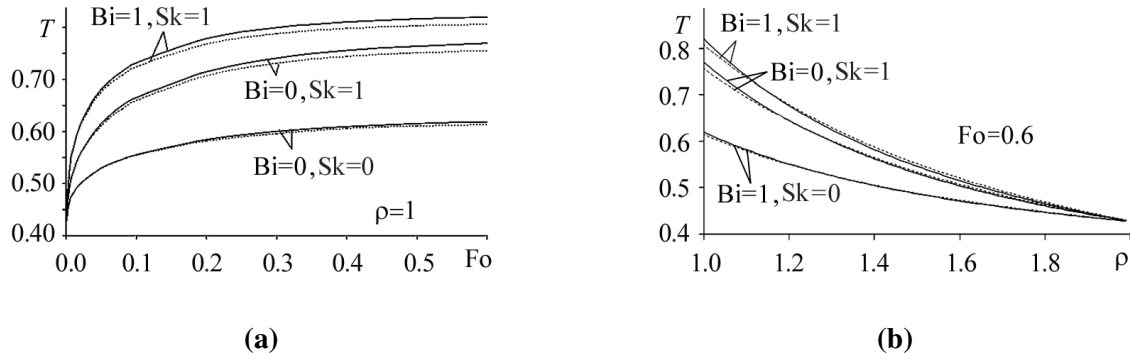


Fig. 2. Distributions of dimensionless temperature in the hollow sphere for different combinations of the modes of heat exchange: (a) depending on Fo on the surface of the cavity, (b) for $Fo = 0.6$ through the thickness of the wall; the solid lines correspond to the dependence of all characteristics of U12 steel on temperature and the dashed lines correspond to constant values of the characteristics.

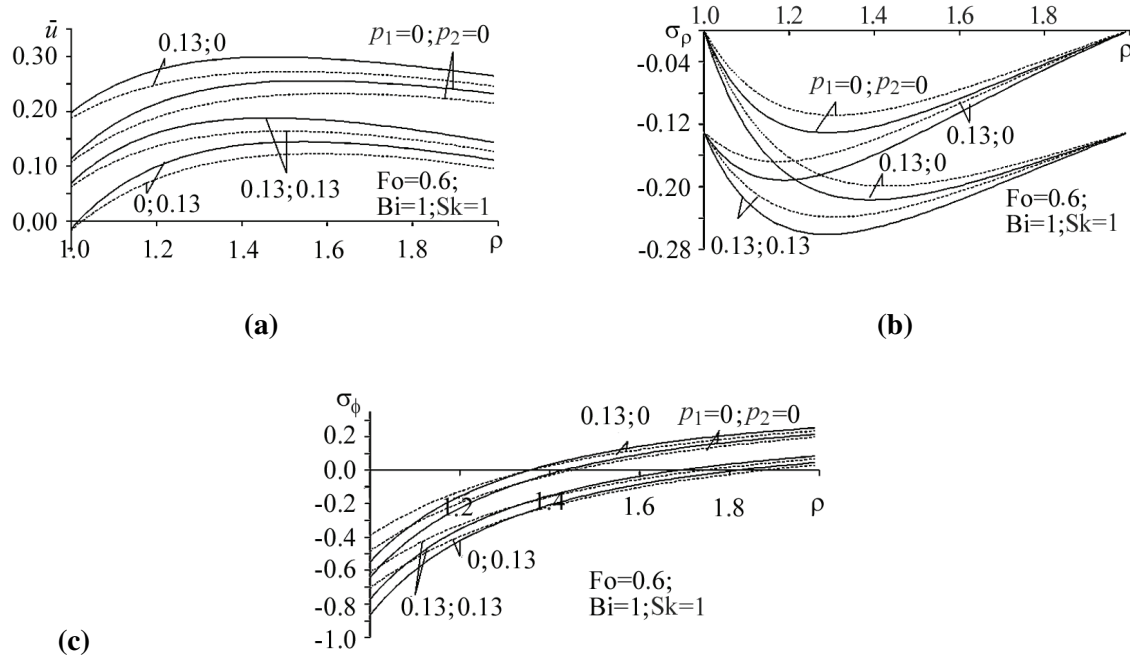


Fig. 3. Distributions of the displacements (a), radial stresses (b), and hoop stresses (c) for $Fo = 0.6$ over the thickness of the wall of the hollow sphere under the conditions of convective-radiant heat exchange for different combinations of pressures on the surfaces; the solid lines correspond to the dependence of all characteristics of U12 steel on temperature and the dashed lines correspond to constant values of the characteristics.

We find the solution of the semidiscrete problem (31) by using the formulas of backward differentiation (by the Geer method for the banded structure of the Jacobian matrix computed as a result of numerical differentiation). On finding the Kirchhoff variable θ_i , the temperature of the hollow sphere is computed according to relation (18) by setting $m = i$. As a result of the limit transition in relation (18) as k approaches zero, we conclude that the temperature of the thermally insensitive hollow sphere at a given point ρ_i and given time Fo is equal to $\theta_i + T_p$. In this case, θ_i is found as a result of the solution of problem (31) with $a^*(\theta_i) = 1$ ($i = 0, 1, \dots, n$).

The corresponding problem of thermoelasticity is solved by the perturbation method. The stress-strain states of the thermosensitive and thermally insensitive spheres are numerically investigated for different combinations of constant pressures $\bar{p}_1$ and $\bar{p}_2$ acting on the inner and outer surfaces, respectively. In the solution of the problem of thermoelasticity for the thermosensitive hollow sphere by the perturbation method, the analysis of the first four computed terms in series (29) and (30) reveals good convergence of these series because the first two terms give more than 95% of the total values of displacements and stresses (Figs. 2 and 3).

The maximum difference between the values of temperature in the thermosensitive and thermally insensitive hollow spheres is observed under the conditions convective-radiant heat exchange through the inner surface and can be as high as 2%.

It follows from Fig. 2a that, as the distance from the internal hollow along the radius increases for $Fo = 0.6$, the displacements in the sphere increase, attain their maximum values (different for different combinations of pressures), and then gradually decrease as the distance to the outer surface decreases. The results of numerical calculations demonstrate that, for $Fo = 0.6$ and a given pressure on the outer surface $\bar{p}_2 = 0.13$, the displacements on the inner surface become negative. The maximum difference between the values of displacements for the thermosensitive and thermally insensitive hollow spheres is about 20%.

It follows from Fig. 3b that the radial stresses along the radius of the hollow sphere are negative for $Fo = 0.6$. It is also shown that the maximum difference between the values of radial stresses for the thermosensitive and thermally insensitive hollow spheres in the absence of loads can be as large as 30%.

As follows from Fig. 3c, the hoop stresses are compressive on the inner surface of the sphere and tensile on its outer surface. Moreover, if the inner surface is loaded, then the hoop stresses change their sign in the region located closer to this surface. At the same time, if the outer surface is loaded, then the sign is changed closer to this surface. In the absence of loads, the maximum difference between the values of hoop stresses for the thermosensitive and thermally insensitive hollow spheres can be as large as 40%.

CONCLUSIONS

It is shown that, in finding the temperature of the hollow sphere, the analysis of the temperature dependences of thermal properties of the material (U12 steel) leads to insignificant difference (2%) between the values of temperature obtained for the thermosensitive and thermally insensitive hollow spheres. At the same time, in the determination of the thermoelastic states of the thermosensitive (with regard for the temperature dependence of the thermal and mechanical characteristics of the material) and thermally insensitive (for constant characteristics equal to their reference values) hollow spheres, the difference between the corresponding values of displacements, radial stresses, and hoop stresses can be as large as 20, 30, 40%, respectively. This reveals the importance of taking into account the temperature dependence of the characteristics of materials in the determination of the thermoelastic state of structural elements under the conditions of high-temperature heating.

REFERENCES

1. V. S. Popovych, O. M. Vovk, and H. Yu. Harmatii, "Thermoelastic state of a thermosensitive sphere under the conditions of complex heat exchange with the environment," *Prikl. Probl. Mekh. Matem.*, Issue 3, 99–106 (2005).
2. V. S. Popovych, O. M. Vovk, and H. Yu. Harmatii, "Thermoelastic state of a thermosensitive layer under the conditions of convective-radiant heating," *Prikl. Probl. Mekh. Matem.*, Issue 2, 171–177 (2004).
3. V. S. Popovych, H. Yu. Harmatii, and O. M. Vovk, "Thermoelastic state of a thermosensitive space with spherical hollow under the conditions of convective-radiant heat exchange," *Matem. Met. Fiz.-Mekh. Polya*, **49**, No. 2, 12–16 (2006).
4. V. S. Popovych and H. Yu. Harmatii, "Stress-strain state of a thermosensitive hollow sphere under the conditions of convective heat exchange with the environment," *Matem. Met. Fiz.-Mekh. Polya*, **48**, No. 1, 146–154 (2005).
5. V. S. Popovych, "Construction of solutions of the problems of thermoelasticity for thermosensitive bodies under the conditions of convective-radiant heat exchange," *Dop. Nats. Akad. Nauk Ukr.*, No. 11, 69–73 (1997).

6. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Clarendon Press, Oxford (1959).
7. Yu. M. Kolyano, *Methods of Heat Conduction and Thermoelasticity for Inhomogeneous Bodies* [in Russian], Naukova Dumka, Kiev (1992).
8. A. V. Lykov, *Theory of Heat Conduction* [in Russian], Vysshaya Shkola, Moscow (1967).
9. K. Dekker and J. G. Verwer, *Stability of Runge–Kutta Method for Stiff Nonlinear Differential Equations*, North-Holland, Amsterdam–New-York–London (1984).
10. V. V. Ivanov, *Numerical Analyses on Computers. Procedures* [in Russian], Naukova Dumka, Kiev (1986).
11. V. A. Lomakin, *Theory of Elasticity of Inhomogeneous Bodies* [in Russian], Moscow University, Moscow (1976).
12. V. S. Popovych and H. T. Sulim, “Centrally symmetric quasistatic problem of thermoelasticity for a thermosensitive body,” *Fiz.-Mekh. Mekh. Mater.*, **40**, No. 3, 62–68 (2004).
13. N. B. Vargaftik (editor), *Thermal Properties of Materials. A Handbook* [in Russian], Gosénergoizdat, Moscow–Leningrad (1956).
14. V. G. Sorokin (editor), *A Handbook of Grades of Steels and Alloys* [in Russian], Mashinostroenie, Moscow (1989).