

ON THE CRITICAL SIZE OF MICROCRACKS INITIATED IN MATERIALS

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We obtain an explicit expression for the potential energy of an elastic body containing a crack with prefracture zones at its tips. This expression is used to construct an analytic dependence of the critical size of a microcrack required for its transformation into a macroscopic defect by using the theory of catastrophes. The reliability of the obtained theoretical values of the critical size of a microcrack is corroborated by comparing these results with the available experimental data.

The problem of determination of the critical size of a microcrack required for its transformation into a macroscopic defect of a material is one of the most important problems in fracture mechanics. A crack is regarded as macroscopic [1, 2] if the total size of the zones of weakened bonds between its lips is much smaller than the total length of the defect. Therefore, it is reasonable to say that cracks in which the length of the region of weakened bonds is greater than the length of the region of free surfaces are microscopic. The notions of micro- and macrocracks often correspond to the notions of stable and unstable crack growth. This fact can be used to determine the critical size of a microcrack required for its transformation into a macrocrack. For this purpose, we use the applied theory of catastrophes [3] to analyze the potential energy of an elastic cracked body computed within the framework of the well-known δ_k -model [1].

Potential Energy of a Body Containing a Crack with Prefracture Zones on Its Continuation

Mathematical computations performed within the framework of the δ_k -model are based on the problem of the theory of elasticity of determination of the stress field in a body containing a crack with prefracture zones on its continuation. Assume that an infinite plate weakened by a crack of length $2l_0$ (Fig. 1) is loaded at infinity by forces with intensity p perpendicular to the crack lips. One can assume that, in the limiting state, the stresses σ_0 induced by the tensile strength of the material under the action of these forces are constant near the crack ends (in the prefracture zones). For the indicated elastic plate, we introduce a coordinate system xOy and direct its x -axis along the crack (Fig. 1). In this case, for the components of stresses, we have $\sigma_y(x, 0) = 0$ for $|x| < l_0$ and $\sigma(x, 0) = \sigma_0$ for $l_0 < |x| < l$, where $l - l_0$ is the length of the prefracture band.

Let us now find the potential energy of this cracked plate. By analogy with the theory of Griffith [4], the potential energy of a body with crack modeled by weakened prefracture zones at its tips can be represented in the form

$$\Pi = \Pi_0 + U(l_0) - W(p, l_0), \quad (1)$$

where Π_0 is the potential energy of the body without crack,

$$U_0(l_0) = 4\gamma l_0, \quad (2)$$

γ is the effective surface energy, and $W(p, l_0)$ is the strain energy. It is easy to see that

$$W(p, l_0) = - \int_{-l}^l \sigma_y(x, 0) v(x, 0) dx = p \int_{-l}^l v(x) dx - \sigma_0 \left(\int_{-l}^{-l_0} v(x) dx + \int_{l_0}^l v(x) dx \right), \quad (3)$$

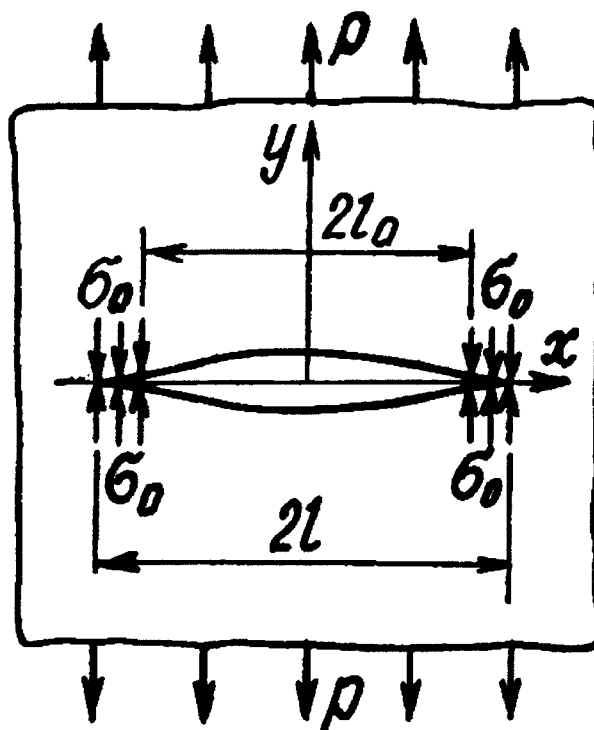


Fig. 1. Schematic diagram of a crack with prefracture zones.

where $v(x)$ is the displacement of the lips of the defect. For each point $x \in [-l, l]$, it can be computed by the formula [1]

$$v(x, 0) = c\sigma_0 \{ (x - l_0) \Gamma(l, x, l_0) - (x + l_0) \Gamma(l, x, -l_0) \}, \quad |x| \leq l, \quad (4)$$

where

$$\Gamma(l, x, \xi) = \ln \frac{l^2 - x\xi - \sqrt{(l^2 - x^2)(l^2 - \xi^2)}}{l^2 - x\xi + \sqrt{(l^2 - x^2)(l^2 - \xi^2)}}, \quad (5)$$

$$c = \frac{1 - \mu^2}{\pi E}$$

under the conditions of plane deformation, $c = (\pi E)^{-1}$ for the plane stressed state, μ is Poisson's ratio, and E is Young's modulus of the material.

By substituting expression (4) in relation (5) and computing the integrals

$$\int \Gamma(l, x, \xi) dx = (x - \xi) \Gamma(l, x, \xi) + 2\sqrt{l^2 - \xi^2} \arccos \frac{x}{l},$$

$$\int x \Gamma(l, x, \xi) dx = \frac{1}{2} (x^2 - \xi^2) \Gamma(l, x, \xi) + \sqrt{(l^2 - x^2)(l^2 - \xi^2)} + \xi \sqrt{l^2 - \xi^2} \arccos \frac{x}{l},$$

we obtain

$$W = 2\pi c l_0 \sigma_0 \sqrt{l^2 - l_0^2} \left(p - \frac{2}{\pi} \sigma_0 \arccos \frac{l}{l_0} \right) - 2\sigma_0^2 l_0^2 c [\Gamma(l, -l_0, l_0) + \Gamma(l, l_0, -l_0)]. \quad (6)$$

At the same time, according to the δ_k -model, we have

$$p = \frac{2}{\pi} \sigma_0 \arccos \frac{l_0}{l}$$

and

$$\frac{l_0}{l} = \sec \frac{\pi p}{2\sigma_0}.$$

Therefore, in view of relation (5), it follows from formula (6) that

$$W = -8\sigma_0^2 l_0^2 c \ln \frac{l_0}{l} = -8\sigma_0^2 l_0^2 c \ln \left(\cos \frac{\pi p}{2\sigma_0} \right). \quad (7)$$

By substituting relations (2) and (7) in formula (1), we arrive at an analytic expression for the potential energy of a body weakened by an actual crack with prefracture zones at its tips:

$$\Pi = \Pi_0 + 4\gamma l_0 + 8\sigma_0^2 l_0^2 c \ln \left(\cos \frac{\pi p}{2\sigma_0} \right). \quad (8)$$

The validity of formula (8) can be proved as follows: If we use the conditions of crack growth proposed by Griffith, i.e.,

$$\left. \frac{\partial \Pi}{\partial l_0} \right|_{p=p_*} = 0 \quad \text{and} \quad \frac{\partial^2 \Pi}{\partial l_0^2} < 0, \quad (9)$$

and differentiate equality (8), then we obtain

$$\gamma = -4\sigma_0^2 l_0 c \ln \left(\cos \frac{\pi p_*}{2\sigma_0} \right). \quad (10)$$

This yields

$$p_* = \frac{2}{\pi} \sigma_0 \arccos \left\{ \exp \left(-\frac{\gamma}{4\sigma_0^2 l_0 c} \right) \right\}. \quad (11)$$

In relations (10) and (11), p_* is the intensity of the critical load.

The second condition in (9) is satisfied automatically. If, in relation (11), we take

$$\gamma = \frac{\delta_c \sigma_0}{2}$$

where δ_c is the critical distance between the noninteracting crack lips, then this relation immediately turns into the well-known formula from [1], namely,

$$p_* = \frac{2}{\pi} \sigma_0 \arccos \left\{ \exp \left(-\frac{\delta_c}{8l_0\sigma_0 c} \right) \right\}. \quad (12)$$

This relation proves the validity of the expression for the potential energy (8).

Analysis of Changes in the Potential Energy of a Cracked Body on the Basis of the Theory of Catastrophes [3]

Unlike the Griffith model, the expression for the potential energy (8) established above enables one to analyze the loss of stability in the process of stable crack growth within the framework of the theory of catastrophes. In this theory, seven types of so-called elementary catastrophes are used in the first approximation for the description of various physical phenomena and related processes [3]. We consider the first of these elementary catastrophes, namely, a folding-type catastrophe. For this purpose, we represent the potential energy (8) in the form

$$\Pi = \Pi_0 + 4\gamma(l_0 + x) + 8\sigma_0^2(l_0 + x)^2 c \ln \frac{l_0 + x}{l}, \quad (13)$$

under the assumption that the length of the crack (its free surface) acquires a relatively small increment x . As a result of computations performed for $x \ll l_0$, we obtain the following approximate expression for the energy Π :

$$\Pi = \frac{8c\sigma_0^2}{l_0} x^3 + 8c\sigma_0 \left(\ln \frac{l_0}{l} + 2 \right) x^2 + \left[4\gamma l_0 + 8c\sigma_0^2 \left(2 \ln \frac{l_0}{l} + 1 \right) l_0 \right] x + \Pi_0 + 4\gamma l_0 + 8c\sigma_0^2 l_0^2 \ln \frac{l_0}{l}. \quad (14)$$

In this expression, we perform the following change of variables:

$$z = x + \frac{B}{3A}, \quad (15)$$

where

$$B = 16c\sigma_0^2 - \delta_c \frac{\sigma_0}{l_0} \quad \text{and} \quad A = 8c \frac{\sigma_0^2}{l_0}. \quad (16)$$

As a result, we arrive at the potential function V of a system admitting a folding-type catastrophe, namely,

$$V = \frac{1}{3} z^3 + Mz + N, \quad (17)$$

where

$$V = \frac{\Pi}{3A},$$

$$M = \frac{C}{3A} - \frac{B}{9A^2},$$

$$N = \frac{D}{3A} - \frac{2B}{81A^3} - \frac{CB}{9A^2},$$

$C = 8c\sigma_0^2 l_0$, and $D = \Pi_0 + \delta_c l_0 \sigma_0$. In the theory of catastrophes, it is customary to say that z is the state variable, M is the controlling parameter, and N is a constant.

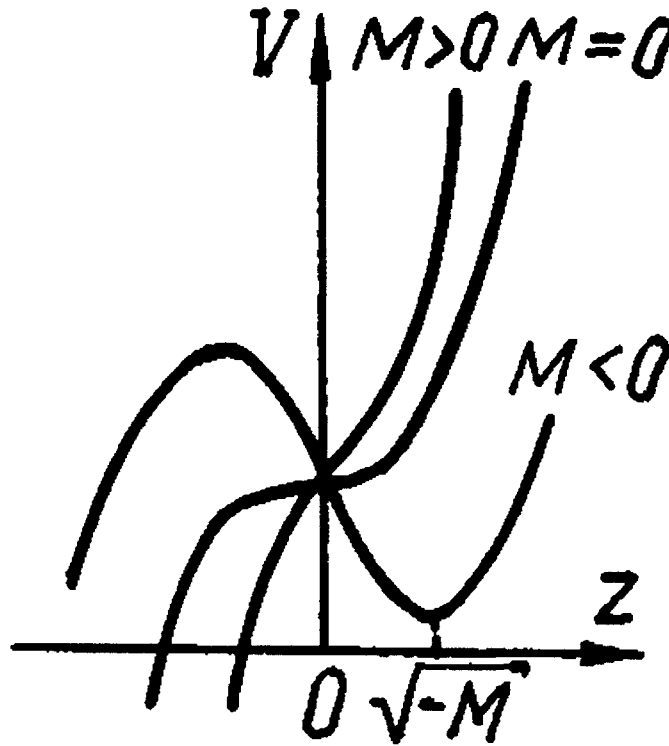


Fig. 2. Changes in the shape of the potential function.

The behavior of functions (17) for different values of the controlling parameter is displayed in Fig. 2. The extrema of the potential function can be found by using the formula

$$\frac{dV}{dz} = z^2 + M = 0. \quad (18)$$

In the theory of catastrophes, this expression is called the manifold of the catastrophe. The stable position of the system is associated with the minimum of the function V attained at

$$z = \sqrt{-M}. \quad (19)$$

Changes in the controlling parameter M are accompanied by smooth changes in the value of the state variable z corresponding to the minimum of V and in the depth of this minimum. The minimum disappears when

$$\frac{d^2V}{dz^2} = 2z = 0. \quad (20)$$

In this case, we observe the appearance of an inflection point corresponding to the case where the points of minimum and maximum coincide. This results in the catastrophe of the function accompanied by an abrupt change in the state z . Hence, the folding-type catastrophe occurs if we have both

$$M = 0 \quad \text{and} \quad z = 0. \quad (21)$$

The comparison of the proposed approximate relation for the potential energy of a body containing a crack weakened by prefracture zones with the potential function from the theory of catastrophes enables one to perform a more accurate analysis of many important problems in linear fracture mechanics than traditional approaches of

mathematical analysis. The application of the theory of catastrophes to fracture mechanics makes it possible, e.g., to determine the critical sizes of microcracks when they become dangerous and lead to the loss of strength of materials.

Determination of the Critical Sizes of Microcracks Formed in Materials

Numerous experiments [5, 6] show that the period of unstable spontaneous crack propagation is almost always preceded by the period of slow critical growth of an initiated crack. Moreover, the crack (e.g., fatigue crack) does not disappear and steadily propagates. It often happens that cracks in brittle materials (such as glass or glass-reinforced plastics) become shorter and merge at their tips. This enables to make a fairly probable assumption that the stability of subcritical crack growth is equivalent to its nonlinearity. At the same time, the investigation of phenomena of this sort becomes successful if they are analyzed within the framework of the δ_k -model and the theory of catastrophes. According to this theory, the point of minimum of the potential function identified with a state of the body depending on the parameter l_0 is fixed in the absence of external perturbations. Under loading, it slowly shifts along the axis of states and, finally, attains the critical values. As is known, both small and large cracks are stable. Therefore, for cracks of all sizes, the potential function must depend on the variation x of the initial length l_0 , i.e., on the quantity $l_0 + x$. On the basis of these arguments, one can use the results presented in the second section to determine the critical size of a microcrack required for its transformation into a macrocrack.

Thus, by using the first equality in (21), we can determine the critical crack length characteristic for a given material, namely,

$$2l_{0cr} = \frac{\delta_c}{8(2 - \sqrt{3})c\sigma_0}, \quad (22)$$

From the second condition, we determine the limiting point of stable evolution of the crack as

$$x_c = \frac{1}{3} \left(2l_{0cr} - \frac{\delta_c}{8c\sigma_0} \right). \quad (23)$$

On the basis of relations (22) and (23), one can estimate the critical sizes of microcracks for different materials and determine the zones of their stable growth.

Example of Estimation of the Critical Size of a Microcrack

Assume that

$$\gamma = \delta_c \frac{\sigma_0}{2} \quad \text{and} \quad \sigma \equiv 0.01E.$$

In this case, for silicate glass ($\gamma = 2.1 \cdot 10^{-8} \text{ J/m}^2$ and $E = 6.7 \cdot 10^2 \text{ N/m}^2$), by using relations (22) and (23), we obtain the following values of the critical length of an initiated crack and the length of the zone of its stable growth: $2l_{0cr} \approx 1.8 \cdot 10^{-6} \text{ m}$ and $x_c \approx 0.5 \cdot 10^{-6} \text{ m}$, respectively. By comparing the established value of the critical length of initiated cracks with experimentally determined lengths of fatigue microcracks [7], we confirm the validity of our calculations. This serves as an additional argument for the applicability of the mathematical theory of catastrophes in fracture mechanics. In exactly the same way, by using the mathematical theory of catastrophes, one can determine the critical sizes of microcracks formed in the process of interaction of cracks with foreign inclusions [8].

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