

METHOD FOR STUDYING FRACTURE OF BODIES WITH A CYLINDRICAL CAVITY AND A COAXIAL CYLINDRICAL CRACK OF FINITE LENGTH UNDER AXIAL COMPRESSION

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The fracture of elastic materials with a cylindrical cavity and a coaxial cylindrical crack near the surface during uniform compression of the medium along its axis is studied using a nonclassical criterion approach associated with a mechanism for local loss of stability near the defect. A solution technique is proposed which makes it possible to reduce the three-dimensional axially symmetric linearized problem to a system of linear homogeneous equations to be solved numerically.

A nonclassical criterion approach [1] associated with a mechanism for local loss of stability near a defect has served as the basis for a general method of solving fracture problems for materials under compression [2, 3]. This method has been used to solve a number of problems for bodies with cracks [4, 5]. In this article this method is used to study the fracture of a body with a cylindrical cavity and a coaxial cylindrical crack near its surface.

Consider an infinite body containing a circular cylindrical cavity of radius b and a cylindrical crack of radius $b + h$ and length $2a$ located on the lateral part of the cylindrical surface $\{r = b + h, 0 \leq \theta < 2\pi, -a \leq x_3 \leq a\}$, whose axis coincides with the cavity axis x_3 and which is compressed along the x_3 axis by uniform forces applied at infinity. Here r, θ, x_3 are cylindrical coordinates and h is the distance between the surface of the circular cylindrical cavity and the surface of the circular cylindrical crack.

We shall solve this mechanical fracture problem in accordance with a common approach for compressible and incompressible elastic bodies with an arbitrary elastic potential, both for large subcritical deformations and for two variants of the theory of small subcritical deformations. We limit ourselves to the case of unequal roots of the characteristic equation [2]. In formulating the linearized problem below, we shall use lagrangian coordinates x_j that coincide with cartesian coordinates in the undeformed state, as well as perturbations in the displacement vector $\vec{u}$ and stress tensor $\mathbf{t}$ relative to unit area of the body in the undeformed state.

The initial forces applied at infinity create a uniform stressed-deformed subcritical state in the neighborhood of the defect, which is determined by the following relationships [2]:

$$\begin{aligned} S_{11}^0 = S_{22}^0 = 0, \quad S_{33}^0 = \text{const} \neq 0, \\ u_j^0 = \delta_{jm}(\lambda_j - 1)x_m, \quad \lambda_j = \text{const}; \\ \lambda_1 = \lambda_2 \neq \lambda_3. \end{aligned} \tag{1}$$

Here S_{ij}^0 are the components of the symmetric stress tensor relative to a unit area of the body in the undeformed state, u_j^0 are the displacements corresponding to the initial stresses S_{ij}^0 , λ_j are the reductions in the lengths along the axes, and δ_{jm} is the Kronecker symbol. For this loading scheme (1), the additional conditions for applicability of a static method for studying the stability of the equilibrium state along the crack are satisfied [2].

We shall assume that the edges of the crack, as well as the surface of the cavity, are free of stresses. Here it is assumed that the surfaces of the defect do not come into a contact interaction. In addition, at the boundary of regions 1 ($b < r < b + h$) and 2 ($b + h < r < \infty$) the continuity conditions for the displacements and stresses must be satisfied outside the crack. Given these remarks, we write the boundary conditions for the linearized problem in the form

$$t_{rr}^{(1)} = 0, \quad t_{r3}^{(1)} = 0 \quad (\rho = \rho_0, -\infty < \zeta < +\infty); \quad (2)$$

$$t_{rr}^{(1)} = t_{rr}^{(2)} = 0, \quad t_{r3}^{(1)} = t_{r3}^{(2)} = 0 \quad (\rho = 1, -\beta < \zeta < \beta); \quad (3)$$

$$u_r^{(1)} = u_r^{(2)}, \quad u_3^{(1)} = u_3^{(2)} \quad (\rho = 1, |\zeta| > \beta); \quad (4)$$

$$t_{rr}^{(1)} = t_{rr}^{(2)}, \quad t_{r3}^{(1)} = t_{r3}^{(2)} \quad (\rho = 1, 0 \leq |\zeta| < \infty). \quad (5)$$

where $\rho = \frac{r}{b+h}$, $\rho_0 = \frac{b}{b+h}$, $\zeta = \frac{x_3}{b+h}$, and $\beta = \frac{a}{b+h}$ are dimensionless quantities and the superscripts 1 and 2 refer to regions 1 and 2.

The boundary conditions, therefore, include conditions at the surface of the cavity (2), conditions at the edges of the cracks (3), and continuity conditions for the displacements (4) and stresses (5) at the interface between the regions.

The linearized equilibrium equations in the undeformed state coordinates [4] are

$$L_{m\alpha} u_\alpha = 0, \quad L_{m\alpha} = \omega_{i\alpha\beta} \frac{\partial^2}{\partial x_i \partial x_\beta} \quad (6)$$

for compressible bodies and

$$L_{m\alpha} u_\alpha + q_{\alpha m} \frac{\partial u_4}{\partial x_\alpha} = 0, \quad q_{m\alpha} \frac{\partial u_\alpha}{\partial x_m} = 0, \quad (7)$$

$$L_{m\alpha} = x_{i\alpha\beta} \frac{\partial^2}{\partial x_i \partial x_\beta}, \quad u_4 \equiv p$$

for incompressible bodies. In the case of uniform subcritical deformations, general solutions can be constructed for these equations. The general solutions of the static linearized equations (6)–(7) for axially symmetric problems in the case of unequal roots ($n_1 \neq n_2$) of the characteristic equation are given in terms of two potential functions which obey the equations

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + n_i \frac{\partial^2}{\partial x_3^2} \right) \varphi_i(r, x_3) = 0 \quad (i = 1, 2) \quad (8)$$

and have the form [4]

$$u_r = \frac{\partial \varphi_1}{\partial r} + \frac{\partial \varphi_2}{\partial r}, \quad u_3 = m_1 \frac{\partial \varphi_1}{\partial x_3} + m_2 \frac{\partial \varphi_2}{\partial x_3}. \quad (9)$$

The components of the perturbations in the Kirchhoff stress tensor t on the cylindrical surface that will be needed below are given by

$$t_{rr} = C_{44} \left[-p_1 \left(l_1 \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial x_3^2} \right) \varphi_1 - p_2 \left(l_2 \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial x_3^2} \right) \varphi_2 \right],$$

$$t_{r3} = C_{44} \left[d_1 \frac{\partial^2 \varphi_1}{\partial r \partial x_3} + d_2 \frac{\partial^2 \varphi_2}{\partial r \partial x_3} \right]. \quad (10)$$

The quantities in Eqs. (9)–(10) depend on the choice of material (form of the elastic potential) and the initial stressed-deformed state. For compressible bodies, they can be written in terms of the components of the tensor ω as

$$\begin{aligned} C_{44} &= \omega_{1313}, \quad m_i = (\omega_{1111}n_i - \omega_{3113})(\omega_{1133} + \omega_{1313})^{-1}, \\ d_i &= 1 + m_i\omega_{1331}\omega_{1313}^{-1}, \quad p_i = (\omega_{1111}n_i + \omega_{1133}m_i)\omega_{1313}^{-1}, \\ l_i &= (\omega_{1111} - \omega_{1122})(\omega_{1111}n_i + \omega_{1133}m_i)^{-1} \quad (i = 1, 2). \end{aligned} \quad (11)$$

For incompressible bodies, they are given in terms of the components of the tensor $\mathfrak{x}$ by

$$\begin{aligned} C_{44} &= \mathfrak{x}_{1313}, \quad m_j = q_{11}q_{33}^{-1}n_j, \quad d_j = 1 + \mathfrak{x}_{1331}\mathfrak{x}_{1313}^{-1}m_j, \\ p_j &= [q_{11}q_{33}^{-1}(\mathfrak{x}_{1133} + \mathfrak{x}_{1313})n_j - \mathfrak{x}_{1133}m_j + \mathfrak{x}_{3113}]\mathfrak{x}_{1313}^{-1}, \\ l_j &= (\mathfrak{x}_{1111} - \mathfrak{x}_{1122})[q_{11}q_{33}^{-1}(\mathfrak{x}_{1133} + \mathfrak{x}_{1313})n_j - \mathfrak{x}_{1133}m_j + \mathfrak{x}_{3113}]^{-1}. \end{aligned} \quad (12)$$

We represent the potential functions $\varphi_1(r, x_3)$ and $\varphi_2(r, x_3)$ separately in each of the regions 1 ($b < r < b + h$) and 2 ($b + h < r < \infty$), applying the subscripts 1 and 2 to them in the corresponding regions, in the form of integral Fourier expansions:

$$\begin{aligned} C_{44}\varphi_i^{(1)} &= \int_0^{+\infty} f_i(\lambda)I_0(\sqrt{n_i}\lambda\rho)\cos\zeta\lambda d\lambda + \int_0^{+\infty} e_i(\lambda)K_0(\sqrt{n_i}\lambda\rho)\cos\zeta\lambda d\lambda, \\ C_{44}\varphi_i^{(2)} &= \int_0^{+\infty} g_i(\lambda)K_0(\sqrt{n_i}\lambda\rho)\cos\zeta\lambda d\lambda \quad (i = 1, 2), \end{aligned} \quad (13)$$

where $f_i(\lambda)$, $e_i(\lambda)$, and $g_i(\lambda)$ ($i = 1, 2$) are unknown weighting functions. I_0 and K_0 are the modified Bessel functions and when there is no cylindrical cavity in the infinite medium, it is necessary to set $e_i(\lambda) \equiv 0$ in the integral representations [5].

Using expansion (13), we obtain the components of displacements (9) and stresses (10) in regions 1 and 2:

$$\begin{aligned} C_{44}(b+h)u_r^{(2)} &= -\sum_{i=1}^2 \int_0^{+\infty} g_i(\lambda)\sqrt{n_i}\lambda K_1(\sqrt{n_i}\lambda\rho)\cos\zeta\lambda d\lambda, \\ C_{44}(b+h)u_3^{(2)} &= -\sum_{i=1}^2 \int_0^{+\infty} g_i(\lambda)m_i\lambda K_0(\sqrt{n_i}\lambda\rho)\sin\zeta\lambda d\lambda, \\ (b+h)^2 t_{r3}^{(2)} &= \sum_{i=1}^2 \int_0^{+\infty} g_i(\lambda)d_i\sqrt{n_i}\lambda^2 K_1(\sqrt{n_i}\lambda\rho)\sin\zeta\lambda d\lambda, \\ (b+h)^2 t_{rr}^{(2)} &= \sum_{i=1}^2 \int_0^{+\infty} g_i(\lambda)p_i\sqrt{n_i}\lambda \left\{ \frac{\lambda}{\sqrt{n_i}}K_0(\sqrt{n_i}\lambda\rho) + \frac{l_i}{\rho}K_1(\sqrt{n_i}\lambda\rho) \right\} \cos\zeta\lambda d\lambda, \end{aligned} \quad (14)$$

and

$$\begin{aligned}
C_{44}(b+h)u_r^{(1)} &= \sum_{i=1}^2 \int_0^{+\infty} \left\{ f_i(\lambda) \sqrt{n_i} \lambda I_1(\sqrt{n_i} \lambda \rho) - e_i(\lambda) \sqrt{n_i} \lambda K_1(\sqrt{n_i} \lambda \rho) \right\} \cos \zeta \lambda d\lambda, \\
C_{44}(b+h)u_3^{(1)} &= - \sum_{i=1}^2 \int_0^{+\infty} \left\{ f_i(\lambda) m_i \lambda I_0(\sqrt{n_i} \lambda \rho) + e_i(\lambda) m_i \lambda K_0(\sqrt{n_i} \lambda \rho) \right\} \sin \zeta \lambda d\lambda, \\
(b+h)^2 t_{r3}^{(1)} &= - \sum_{i=1}^2 \int_0^{+\infty} \left\{ f_i(\lambda) d_i \sqrt{n_i} \lambda^2 I_1(\sqrt{n_i} \lambda \rho) - e_i(\lambda) d_i \sqrt{n_i} \lambda^2 K_1(\sqrt{n_i} \lambda \rho) \right\} \sin \zeta \lambda d\lambda, \\
(b+h)^2 t_{rr}^{(1)} &= \sum_{i=1}^2 \int_0^{+\infty} \left[f_i(\lambda) p_i \sqrt{n_i} \lambda \left\{ \frac{\lambda}{\sqrt{n_i}} I_0(\sqrt{n_i} \lambda \rho) - \frac{l_i}{\rho} I_1(\sqrt{n_i} \lambda \rho) \right\} + \right. \\
&\quad \left. + e_i(\lambda) p_i \sqrt{n_i} \lambda \left\{ \frac{\lambda}{\sqrt{n_i}} K_0(\sqrt{n_i} \lambda \rho) + \frac{l_i}{\rho} K_1(\sqrt{n_i} \lambda \rho) \right\} \right] \cos \zeta \lambda d\lambda. \tag{15}
\end{aligned}$$

Satisfying boundary conditions (2)–(5) and taking Eqs. (14) and (15) into account, we find a system of paired integral equations of the following form:

$$\begin{aligned}
\int_0^{+\infty} \lambda \left\{ [\alpha_{11} + \bar{\psi}_{11}(\lambda)] D_r(\lambda) + \bar{\psi}_{12}(\lambda) D_3(\lambda) \right\} \cos \zeta \lambda d\lambda &= 0 \quad (0 \leq \zeta < \beta), \\
\int_0^{+\infty} \lambda \left\{ [\alpha_{22} + \bar{\psi}_{22}(\lambda)] D_3(\lambda) + \bar{\psi}_{21}(\lambda) D_r(\lambda) \right\} \sin \zeta \lambda d\lambda &= 0 \quad (0 \leq \zeta < \beta) \tag{16}
\end{aligned}$$

and

$$\int_0^{+\infty} D_r(\lambda) \cos \zeta \lambda d\lambda = 0, \quad \int_0^{+\infty} D_3(\lambda) \sin \zeta \lambda d\lambda = 0 \quad (\zeta > \beta). \tag{17}$$

Here

$$\bar{\psi}_{11}(\lambda) = \bar{\psi}_{22}(\lambda) = O(1/\lambda^2), \quad \bar{\psi}_{12}(\lambda) = \bar{\psi}_{21}(\lambda) = O(1/\lambda);$$

$D_r(\lambda)$ and $D_3(\lambda)$ are unknown functions; α_{11} and α_{22} are constants.

In solving the system of paired integral equations, we shall choose a solution of a form such that the integral equations specified over the interval $\beta < \zeta < +\infty$ are satisfied identically. The remaining equations are reduced to a system of algebraic equations. To do this, we shall seek unknown functions $D_r(\lambda)$ and $D_3(\lambda)$ in the form of series in Bessel functions of the first kind

$$D_r(\lambda) = \sum_{j=0}^{+\infty} a_j \lambda^{-1} J_{2j+1}(\beta \lambda), \quad D_3(\lambda) = \sum_{j=0}^{+\infty} b_j \lambda^{-1} J_{2j+2}(\beta \lambda) \tag{18}$$

with unknown constants a_j and b_j [6].

Direct substitution of expansions (18) in Eqs. (17) causes them to be satisfied identically [7], while Eqs. (16) take the form

$$\int_0^{+\infty} \left\{ \sum_{j=0}^{+\infty} a_j [\alpha_{11} + \bar{\psi}_{11}(\lambda)] J_{2j+1}(\beta\lambda) + \sum_{j=0}^{+\infty} b_j \bar{\psi}_{12}(\lambda) J_{2j+2}(\beta\lambda) \right\} \cos \varsigma \lambda \, d\lambda = 0,$$

$$\int_0^{+\infty} \left\{ \sum_{j=0}^{+\infty} a_j \bar{\psi}_{21}(\lambda) J_{2j+1}(\beta\lambda) + \sum_{j=0}^{+\infty} b_j [\alpha_{22} + \bar{\psi}_{22}(\lambda)] J_{2j+2}(\beta\lambda) \right\} \sin \varsigma \lambda \, d\lambda = 0 \quad (0 \leq \varsigma < \beta). \quad (19)$$

Then using a representation of the functions $\cos \varsigma \lambda$ and $\sin \varsigma \lambda$ in the form of the series [7]

$$\cos \varsigma \lambda = \sum_{i=0}^{+\infty} \varepsilon_i J_{2i}(\beta\lambda) \cos 2i\varphi, \quad \varepsilon_0 = 1, \quad \varepsilon_i = 2 \quad (i \geq 1);$$

$$\sin \varsigma \lambda = \sum_{i=0}^{+\infty} 2 J_{2i+1}(\beta\lambda) \sin (2i+1)\varphi, \quad \varphi = \arcsin(\varsigma/\beta) \quad (20)$$

and equating the coefficients of the harmonics in Eqs. (19) to zero, we obtain the following system of linear algebraic equations in the unknowns a_j and b_j :

$$\sum_{j=0}^{+\infty} [\alpha_{11} P_{ij}^0(\beta\lambda) + P_{ij}(\beta\lambda)] a_j + \sum_{j=0}^{+\infty} Q_{ij}(\beta\lambda) b_j = 0,$$

$$\sum_{j=0}^{+\infty} R_{ij}(\beta\lambda) a_j + \sum_{j=0}^{+\infty} [\alpha_{22} T_{ij}^0(\beta\lambda) + T_{ij}(\beta\lambda)] b_j = 0 \quad (i = 0, 1, 2, \dots), \quad (21)$$

where

$$P_{ij}(\beta\lambda) = \int_0^{+\infty} \bar{\psi}_{11}(\lambda) J_{2i}(\beta\lambda) J_{2j+1}(\beta\lambda) d\lambda,$$

$$Q_{ij}(\beta\lambda) = \int_0^{+\infty} \bar{\psi}_{12}(\lambda) J_{2i}(\beta\lambda) J_{2j+2}(\beta\lambda) d\lambda,$$

$$R_{ij}(\beta\lambda) = \int_0^{+\infty} \bar{\psi}_{21}(\lambda) J_{2i+1}(\beta\lambda) J_{2j+1}(\beta\lambda) d\lambda,$$

$$T_{ij}(\beta\lambda) = \int_0^{+\infty} \bar{\psi}_{22}(\lambda) J_{2i+1}(\beta\lambda) J_{2j+2}(\beta\lambda) d\lambda \quad (22)$$

and

$$P_{ij}^0(\beta\lambda) = \int_0^{+\infty} J_{2i}(\beta\lambda) J_{2j+1}(\beta\lambda) d\lambda = T_{ij}^0(\beta\lambda) = \int_0^{+\infty} J_{2i+1}(\beta\lambda) J_{2j+2}(\beta\lambda) d\lambda = \begin{cases} \frac{(-1)^{j-i}}{2\beta}, & i \leq j, \\ \frac{(-1)^{i-j-1}}{2\beta}, & i > j. \end{cases} \quad (23)$$

We have, therefore, reduced the linearized problem formulated above to an eigenvalue problem for the system of equations (21) which can be solved numerically.

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