

INTEGRAL EQUATIONS OF A PLANE PROBLEM OF THE ELASTICITY THEORY FOR A MULTIPLY CONNECTED QUASIORTHOTROPIC BODY

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We construct a system of singular integral equations for the first fundamental problem of the plane elasticity theory for a quasiorthotropic body containing holes and cracks. For this purpose, we use the well-known integral equations obtained for a system of curvilinear cracks (cuts) in a quasiorthotropic plane. The integral equations for a multiply connected region with holes are constructed with the help of the limit transition from open cuts in an infinite elastic plane to closed cuts. These singular integral equations of the first kind on closed contours (boundary of the body) are supplemented with the corresponding regularizing functionals guaranteeing the unique solvability of the integral equations for arbitrary right-hand sides.

Keywords: plane problem of the elasticity theory, quasiorthotropic body, multiply connected region, holes and curvilinear cracks, method of singular integral equations.

Composite materials are extensively used in the contemporary engineering. Their stress-strain state is described by the elasticity theory of orthotropic bodies. The degenerate orthotropic material whose shear modulus is connected with the other characteristics of the material by a relationship leading to multiple roots of the characteristic equation is called quasiorthotropic [1–6]. This class of materials also includes isotropic bodies. In the literature, these materials are also called special orthotropic [7–10], degenerate orthotropic [11, 12], and pseudoisotropic [13, 14]. It was discovered [2, 6] that the ratio of the moduli of elasticity is the main mechanical parameter for orthotropic materials. In other words, a large number of orthotropic materials can be approximately regarded as quasiorthotropic with the same ratio of the moduli of elasticity.

In what follows, we construct a system of singular integral equations for a plane problem of the theory of elasticity for a finite multiply connected quasiorthotropic domain with curvilinear holes and cracks in the case where the stresses are specified on the boundary of the body. We use the well-known integral equations [4] for a system of curvilinear cracks in the quasiorthotropic plane. The cracks (cuts) are regarded as closed. For a multiply connected domain with holes, we construct the integral equations and solve these equations only under additional (equilibrium) conditions that should be satisfied by the right-hand sides of these equations. For the unconditional solvability, we add certain functionals equal to zero under the equilibrium conditions to the left-hand sides of these equations.

The integral equations modified in this way are uniquely solvable for any right-hand side, which allows us to find their solutions by numerical methods [15].

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Basic Relations of the Theory of Elasticity for Quasiorthotropic Bodies

The linear relations for the components of the stress tensor σ_x , σ_y , and τ_{xy} and the strain tensor ε_x , ε_y , and ε_{xy} (Hooke's law) in the plane stressed state for an orthotropic body, where the coordinate axes x and y are directed along the principal axes of orthotropy, take the form [16]

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y, \quad \varepsilon_y = a_{12}\sigma_x + a_{22}\sigma_y, \quad 2\varepsilon_{xy} = a_{66}\tau_{xy}. \quad (1)$$

Here,

$$a_{11} = 1/E_x, \quad a_{22} = 1/E_y, \quad a_{66} = 1/G, \quad a_{12} = -\nu_{xy}/E_x,$$

$E_x = E_1$ and $E_y = E_2$ ($E_x = E_2$ and $E_y = E_1$) are the moduli of elasticity along the x and y axes, respectively, $\nu_{xy} = \nu_{12}$ ($\nu_{xy} = \nu_{21} = \nu_{12}E_2/E_1$) is Poisson's ratio, and G is the shear modulus.

For the case of plane deformation, the elastic constants a_{ij} in Hooke's law (1) should be replaced by the quantities

$$a'_{ij} = a_{ij} - (a_{i3}a_{j3})/a_{33},$$

where $a_{13} = -\nu_{13}/E_1$, $a_{23} = -\nu_{23}/E_2$, and $a_{33} = 1/E_3$ are the corresponding elastic characteristics of the orthotropic material.

For the quasiorthotropic material, the constants a_{ij} satisfy the relation

$$a_{66} = 2\left(\sqrt{a_{11}a_{22}} - a_{12}\right),$$

which takes the following form for the plane stressed state:

$$G = \frac{E_1}{2\left(\sqrt{E_1/E_2} + \nu_{12}\right)}.$$

We now introduce the stress function $F(x, y)$ by the formulas [16]

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}. \quad (2)$$

From the condition of compatibility of strains without mass forces, we get the following elliptic differential equation of the fourth order for the stress function $F(x, y)$ in a quasiorthotropic body:

$$\frac{\partial^4 F}{\partial y^4} + 2\gamma^2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \gamma^4 \frac{\partial^4 F}{\partial x^4} = 0. \quad (3)$$

Its characteristic equation takes the form

$$\mu^4 + 2\gamma^2\mu^2 + \gamma^4 = 0. \quad (4)$$

Here,

$$\gamma = \sqrt[4]{a_{22}/a_{11}}$$

is the parameter of orthotropy (for the plane stressed state, $\gamma = \sqrt[4]{E_x/E_y}$). Equation (4) has complex conjugate multiple roots

$$\mu_1 = \mu_2 = i\gamma, \quad \bar{\mu}_1 = \bar{\mu}_2 = -i\gamma.$$

For $\gamma = 1$, we get the case of an isotropic material.

The general solution of Eq. (3) for the quasiorthotropic body can be represented in terms of analytic functions $\Phi_1(z_1)$ and $\chi_1(z_1)$ of the complex argument $z_1 = x + i\gamma y$ in the form [16]

$$F(x, y) = \operatorname{Re}[\bar{z}_1\Phi_1(z_1) + \chi_1(z_1)]. \quad (5)$$

Relations (2) and (5) yield the following expressions for the components of stresses in terms of the complex potentials $\Phi_1(z_1) = \varphi_1'(z_1)$ and $\Psi_1(z_1) = \psi_1'(z_1) = \chi_1''(z_1)$:

$$\begin{cases} \sigma_y + \gamma^{-2}\sigma_x = 4 \operatorname{Re} \Phi_1(z_1), \\ \sigma_y - \gamma^{-2}\sigma_x + 2i\gamma^{-1}\tau_{xy} = 2[\bar{z}_1\Phi_1'(z_1) + \Psi_1(z_1)]. \end{cases}$$

The components of the vector of displacements u and v can also be represented via the complex potentials $\Phi_1(z_1)$ and $\Psi_1(z_1) = \chi_1'(z_1)$ [17]:

$$2G \left[u + \left(\frac{i}{\gamma} \right) v \right] = \kappa \Phi_1(z_1) - z_1 \overline{\Phi_1'(z_1)} - \overline{\Psi_1(z_1)}, \quad (6)$$

where

$$\kappa = \frac{3\sqrt{a_{22}/a_{11}} + a_{12}/a_{22}}{\sqrt{a_{22}/a_{11}} - a_{12}/a_{22}}$$

($\kappa = (3\gamma^2 - \nu_{xy})/(\gamma^2 + \nu_{xy})$ for the plane stressed state).

By using equality (6), we find the derivatives of displacements as follows [4]:

$$2G \frac{d}{dt_1} \left(u + \left(\frac{i}{\gamma} \right) v \right) = \kappa \Phi_1(t_1) - \overline{\Phi_1(t_1)} - \frac{d\bar{t}_1}{dt_1} (t_1 \overline{\Phi_1'(t_1)} + \overline{\Psi_1(t_1)}), \quad t_1 \in L_1, \quad (7)$$

where L_1 is a contour in the auxiliary plane $z_1 = x + i\gamma y$ corresponding to a curvilinear contour L in the complex plane $z = x + iy$.

Let X_n and Y_n be Cartesian components of the vector of stresses acting along the positive normal n on the curvilinear contour L . They are connected with the normal and tangential components N and T of the stresses by the formula [18]

$$X_n + iY_n = -i \frac{dt}{ds} (N + iT) = \frac{d}{ds} \left(\frac{\partial F}{\partial y} - i \frac{\partial F}{\partial x} \right), \quad (8)$$

where s is the arc abscissa of a point $t = x + iy \in L$.

By using representations (5) and (18), we find [4]

$$\left(\left(\frac{i}{\gamma} \right) X_n - Y_n \right) \frac{ds}{dt_1} = \Phi_1(t_1) + \overline{\Phi_1(t_1)} + \frac{d\bar{t}_1}{dt_1} \left(t_1 \overline{\Phi_1'(t_1)} + \overline{\Psi_1(t_1)} \right), \quad t_1 \in L_1. \quad (9)$$

Relations (6), (7), and (9) make it possible to reduce the principal problems of the elasticity theory for multiply connected quasiorthotropic bodies to boundary-value problems of the theory of functions of complex variable.

By using relation (9), we find the components X and Y of the principal vector of forces acting on the contour L :

$$\begin{aligned} \left(\frac{i}{\gamma} \right) X - Y &= \int_L \left(\left(\frac{i}{\gamma} \right) X_n - Y_n \right) ds \\ &= \int_{L_1} \left[\Phi_1(t_1) + \overline{\Phi_1(t_1)} + \frac{d\bar{t}_1}{dt_1} \left(t_1 \overline{\Phi_1'(t_1)} + \overline{\Psi_1(t_1)} \right) \right] dt_1. \end{aligned} \quad (10)$$

A similar formula for the principal moment M of these forces relative to the origin takes the form:

$$\begin{aligned} M &= \int_L (xY_n - yX_n) ds = -\operatorname{Re} \int_{L_1} \left\{ \left[\Phi_1(t_1) + \overline{\Phi_1(t_1)} + \frac{d\bar{t}_1}{dt_1} \left(t_1 \overline{\Phi_1'(t_1)} + \overline{\Psi_1(t_1)} \right) \right] \bar{t}_1 \right\} dt_1 \\ &= -\operatorname{Re} \int_{L_1} t_1 \Psi_1(t_1) dt_1. \end{aligned} \quad (11)$$

Hence, the rotation ε of a point z of the elastic domain is given by the formula

$$\varepsilon = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{\gamma(\kappa + 1)}{2G} \operatorname{Im} [\Phi_1(z_1)]. \quad (12)$$

If the elastic quasiorthotropic plane suffers the action of biaxial tension by stresses $\sigma_y^\infty = p$ and $\sigma_x^\infty = q$ and shear $\tau_{xy}^\infty = \tau$ at infinity, then its homogeneous stressed state is described by the potentials

$$\Phi_1^0(z_1) = \frac{p + \gamma^{-2}q}{4}, \quad \Psi_1^0(z_1) = \frac{p - \gamma^{-2}q}{2} + i\gamma^{-1}\tau, \quad (13)$$

and the vector of normal (N) and tangential (T) stresses on the curvilinear contour L is given by the formula

$$N(t) + iT(t) = p_0(t) = \frac{1}{2}(p + q) + \frac{1}{2}(q - p - 2i\tau) \frac{d\bar{t}}{dt}, \quad t = x + iy \in L,$$

which is independent of the elastic constants of the material and remains unchanged for the isotropic plane.

Curvilinear Crack [4]

Consider the first basic problem of the plane elasticity theory for a quasiorthotropic body with curvilinear crack (cut) along a smooth contour L in the case where balanced stresses

$$N^+ + iT^+ = N^- + iT^- = p(t), \quad t \in L, \quad (14)$$

vanishing at infinity, are given on the edges of the cut. Here and in what follows, the superscripts “+” and “−” indicate the limiting values of the corresponding quantities as $z \rightarrow t \in L$ from the left (+) or from the right (−), respectively, relative to the chosen positive direction of traversing the contour L .

We now use the integral representations of the complex potentials

$$\Phi_1(z_1) = \frac{1}{2\pi} \int_{L_1} \frac{g'_1(t_1) dt_1}{t_1 - z_1}, \quad \Psi_1(z_1) = \frac{1}{2\pi} \int_{L_1} \left[\frac{\overline{g'_1(t_1)} d\bar{t}_1}{t_1 - z_1} - \frac{\bar{t}_1 g'_1(t_1) dt_1}{(t_1 - z_1)^2} \right], \quad (15)$$

where the unknown function

$$g'_1(t_1) = \frac{E_x}{4i\gamma^2} \frac{d}{dt_1} \left[\left(u + \left(\frac{i}{\gamma} \right) v \right)^+ - \left(u + \left(\frac{i}{\gamma} \right) v \right)^- \right]$$

is expressed in terms of the jump of displacements on the contour L .

The boundary condition (14) takes the form

$$\begin{aligned} \left[\left(\frac{i}{\gamma} \right) X_n^\pm - Y_n^\pm \right] \frac{ds}{dt_1} &= \tilde{P}(t) = \tilde{P}_1(t_1) \\ &= \frac{1}{2\gamma} \left[(1 + \gamma)p(t) - (1 - \gamma)\overline{p(t)} \frac{d\bar{t}}{dt} \right] \frac{dt}{dt_1}. \end{aligned} \quad (16)$$

Satisfying this condition with the help of potentials (15), we get a singular integral equation for the unknown function $g'_1(t_1)$:

$$\frac{1}{\pi} \int_{L_1} [K_1(\tau_1, t_1) g'_1(\tau_1) d\tau_1 + L_1(\tau_1, t_1) \overline{g'_1(\tau_1)} d\bar{\tau}_1] = \tilde{P}_1(t_1), \quad (17)$$

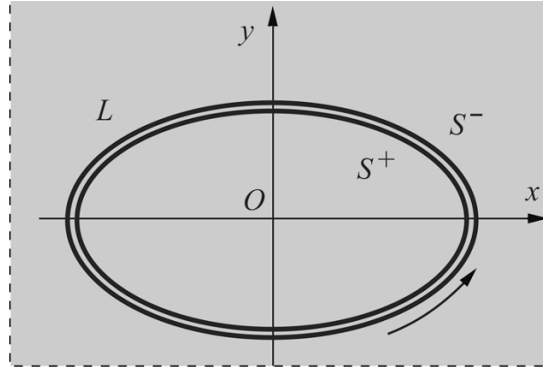


Fig. 1. Closed curvilinear cut in an elastic quasiorthotropic plane.

where

$$\begin{aligned} K_1(\tau_1, t_1) &= \frac{1}{2} \left[\frac{1}{\tau_1 - t_1} + \frac{1}{\bar{\tau}_1 - \bar{t}_1} \frac{d\bar{t}_1}{dt_1} \right], \\ L_1(\tau_1, t_1) &= \frac{1}{2} \left[\frac{1}{\bar{\tau}_1 - \bar{t}_1} - \frac{\tau_1 - t_1}{(\bar{\tau}_1 - \bar{t}_1)^2} \frac{d\bar{t}_1}{dt_1} \right]. \end{aligned} \quad (18)$$

The solution of Eq. (17) must satisfy the condition

$$\int_{L_1} g'_1(t_1) dt_1 = 0 \quad (19)$$

guaranteeing the uniqueness of displacements in traversing the crack contour L .

The complex stress potentials (15) and the singular integral equations (17) remain valid for a system of curvilinear cracks in the quasiorthotropic plane, where the symbol L denotes the collection of crack contours. In this case, the additional condition of uniqueness of displacements (19) should be satisfied for each crack separately.

Closed Curvilinear Cut

In the quasiorthotropic plane, we now consider a closed cut L whose edges suffer the action of the self-balanced load (14). Relations (15) and (17) yield the integral representation and the singular integral equations for the inner (the domain S^+ is a curvilinear disk) and outer (the domain S^- is an infinite plane with curvilinear hole) first basic problems of the elasticity theory (Fig. 1).

In view of relations (16), formulas (10) and (11) yield the following expressions for the components of the principal vector:

$$\left(\frac{i}{\gamma} \right) X - Y = \int_{L_1} \tilde{P}_1(t_1) dt_1 = \int_L p(t) dt \quad (20)$$

and the principal moment

$$M = -\operatorname{Re} \int_{L_1} \tilde{P}_1(t_1) \bar{t}_1 dt_1 = -\operatorname{Re} \int_L p(t) \bar{t} dt \quad (21)$$

of forces acting on the side of the positive (outer) normal n on the contour L .

Comparing the complex potentials (15), integral equation (17), its kernels (18), and the formulas for the principal vector (20) and principal moment (21) with the corresponding relations obtained for the isotropic case [15, 19], we immediately reveal a similarity of these problems. Thus, in order to get the equations for the quasiorthotropic elastic domain from the corresponding relations for the isotropic body, it is necessary to replace the function $p(t)$ with $\tilde{P}_1(t_1)$ and add the index “1” in the corresponding notations. In what follows, we use this analogy in order to construct modified integral equations for a multiply connected domain weakened by holes and cracks.

Substituting potentials (15) in relations (10) and (11), we get the following conditions:

$$\int_{L_1} \tilde{P}_1(t_1) dt_1 = 0, \quad \operatorname{Re} \int_{L_1} \tilde{P}_1(t_1) \bar{t}_1 dt_1 = 0.$$

In other words, these potentials specify the stress-strain state in the domains S^+ and S^- in the case where a balanced load acts upon the boundary contour L . Hence, the integral equation (17) possesses a solution on the closed contour only in the case where its right-hand side satisfies the equilibrium condition. For the efficient numerical solution of the problem, it is necessary to modify this equation in order to guarantee that the modified equation is uniquely solvable for any right-hand side.

Adding certain operators (equal to zero if the equilibrium conditions are satisfied) to the right-hand side of Eq. (17), we get the following modified singular integral equation on the closed contour:

$$\int_{L_1} [K_1(\tau_1, t_1) g'_1(\tau_1) d\tau_1 + L_1(\tau_1, t_1) \overline{g'_1(\tau_1)} d\bar{\tau}_1] - \frac{M^{(1)}}{2i\bar{t}_1^2} \frac{d\bar{t}_1}{dt_1} + \frac{a_0}{l} \frac{ds_1}{dt_1} = \pi \tilde{P}_1(t_1), \quad t_1 \in L_1. \quad (22)$$

This equation is uniquely solvable for any right-hand side [15]. Here, s_1 is the arc abscissa corresponding to the point t_1 , l is an arbitrary parameter with the dimension of length, and

$$M^{(1)} = i \int_{L_1} [g'_1(t_1) \bar{t}_1 dt_1 - \overline{g'_1(t_1)} t_1 d\bar{t}_1], \quad a_0 = \int_{L_1} g'_1(t_1) dt_1.$$

In this case, it is accepted that the origin lies in the domain S^+ .

The moment $M^{(1)}$ in relation (22) can be represented in terms of the required function. Moreover, it can be replaced by the quantity

$$M^{(0)} = il^2 \int_{L_1} \left[g'_1(t_1) \frac{dt_1}{t_1} - \overline{g'_1(t_1)} \frac{d\bar{t}_1}{\bar{t}_1} \right]$$

proportional to a rotation of the origin [see relation (12)].

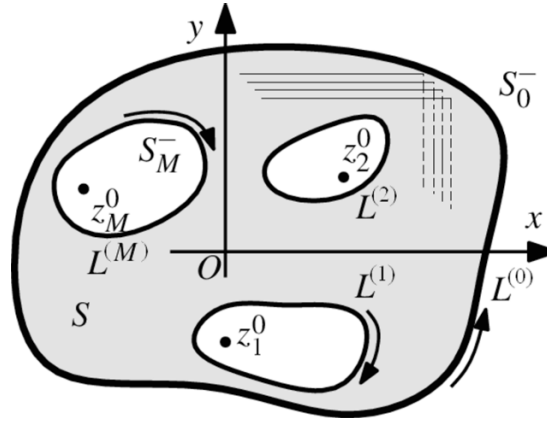


Fig. 2. Finite elastic quasiorthotropic region with holes.

System of Curvilinear Holes in a Bounded Body

We now pass to the general analysis of the first basic problem of the theory of elasticity for a multiply connected quasiorthotropic domain with curvilinear holes. Assume that the domain S occupied by an elastic body is bounded by one or several closed contours $L^{(1)}, L^{(2)}, \dots, L^{(M)}, L^{(0)}$, where the first M contours are placed near each other, and the last contour encloses the remaining contours. The positive direction of traversing the contours $L^{(k)}$ ($k = 1, 2, \dots, M$) is chosen so that the domain S remains on the left (Fig. 2).

Consider the first basic problem. Assume that the stresses

$$N + iT = p(t), \quad t \in L = \bigcup L^{(k)} \quad (k = \overline{0, M}), \quad (23)$$

satisfying the balance conditions are specified on the contours $L^{(k)}$. This means that the principal vector and the principal moment of external forces acting on the boundary of the body are equal to zero:

$$\sum_{k=0}^M \int_{L^{(k)}} p(t) dt = 0, \quad \sum_{k=0}^M \int_{L^{(k)}} \bar{t} p(t) dt = 0. \quad (24)$$

The complex potentials for a nonself-balanced load on the internal contours $L^{(k)}$ ($k = \overline{0, M}$) can be represented in the form

$$\Phi_1^*(z_1) = -\frac{1}{2\pi\gamma(1+\kappa)} \sum_{k=1}^M \frac{X_k + i\gamma Y_k}{z_1 - \bar{z}_k^0} + \Phi_1(z_1),$$

$$\Psi_1^*(z_1) = \frac{1}{2\pi\gamma(1+\kappa)} \sum_{k=1}^M \left[\frac{\kappa(X_k - i\gamma Y_k)}{z_1 - \bar{z}_k^0} - \frac{\bar{z}_k^0(X_k + i\gamma Y_k)}{(z_1 - \bar{z}_k^0)^2} \right] + \Psi_1(z_1),$$

where X_k and Y_k are the components of the principal vector of external forces acting on the contour $L^{(k)}$,

$$\tilde{z}_k^0 = \frac{(1+\gamma)z_k^0 + (1-\gamma)\bar{z}_k^0}{2},$$

z_k^0 is an arbitrary point inside the contour $L^{(k)}$, and the functions $\Phi_1(z_1)$ and $\Psi_1(z_1)$ are analytic in the domain S and correspond to the stressed state in which the principal vectors of external forces acting on the contours $L^{(k)}$ ($k = \overline{0, M}$) are equal to zero. Thus, for the multiply connected domain containing M holes with boundary contours $L^{(k)}$ ($k = \overline{0, M}$), we seek the complex potentials $\Phi_1(z_1)$ and $\Psi_1(z_1)$ in the form

$$\begin{aligned}\Phi_1(z_1) &= \frac{1}{2\pi} \int_{L_1} \frac{g'_1(t_1) dt_1}{t_1 - z_1}, \\ \Psi_1(z_1) &= \frac{1}{2\pi i} \sum_{k=1}^M \frac{M^{(k)}}{(z_1 - \tilde{z}_k^0)^2} + \frac{1}{2\pi} \int_{L_1} \left[\frac{\overline{g'_1(t_1)} d\bar{t}_1}{t_1 - z_1} - \frac{\bar{t}_1 g'_1(t_1) dt_1}{(t_1 - z_1)^2} \right],\end{aligned}\tag{25}$$

where $L_1 = \bigcup L_1^{(k)}$ ($k = \overline{0, M}$), and the quantities $M^{(k)}$ are given by the formula

$$M^{(k)} = i \int_{L_1^{(k)}} \left[g'_1(t_1) \bar{t}_1 dt_1 - \overline{g'_1(t_1)} t_1 d\bar{t}_1 \right] \quad (k = 1, 2, \dots, M).$$

Substituting potentials (25) in the boundary conditions (23), we arrive at the system of $M+1$ singular integral equations

$$\begin{aligned}& \int_{L_1} [K_1(\tau_1, t_1) g'_1(\tau_1) d\tau_1 + L_1(\tau_1, t_1) \overline{g'_1(\tau_1)} d\bar{\tau}_1] \\ & - \frac{1}{2i} \sum_{k=0}^M \frac{M^{(k)}}{(\bar{t}_1 - \tilde{z}_k^0)^2} \frac{d\bar{t}_1}{dt_1} + \frac{a_n^0}{l} \frac{ds_1}{dt_1} = \pi \tilde{P}_1(t_1), \quad t_1 \in L_1^{(n)}, \quad n = \overline{0, M},\end{aligned}\tag{26}$$

for $M+1$ functions $g'_1(t_1)$ ($t_1 \in L_1^{(k)}$, $k = 0, 1, \dots, M$). Here, the kernels $K_1(\tau_1, t_1)$ and $L_1(\tau_1, t_1)$ are determined from relations (18) and $\tilde{z}_0^0 = z_0^0 = 0 \in S$ (Fig. 2). To the left-hand side of system (26), we add certain functionals

$$\begin{aligned}M^{(0)} &= il^2 \sum_{k=0}^M \int_{L_1^{(k)}} \left[g'_1(t_1) \frac{dt_1}{t_1} - \overline{g'_1(t_1)} \frac{d\bar{t}_1}{\bar{t}_1} \right], \\ a_n^0 &= \int_{L_1^{(n)}} g'_1(t_1) dt_1, \quad n = 0, 1, \dots, M,\end{aligned}\tag{27}$$

equal to zero if the balance conditions (24) are satisfied and guaranteeing the absence of rotation at the point $z=0$ and the uniqueness of displacements in traversing the contours of the holes. The system of integral equations (26) is uniquely solvable for any right-hand side $\tilde{P}_1(t_1)$.

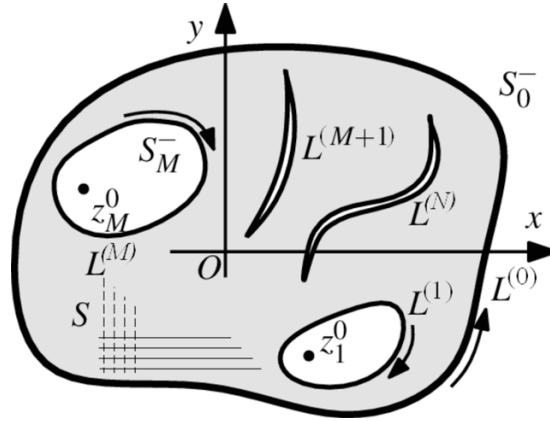


Fig. 3. Elastic quasiorthotropic region with holes and cracks.

Multiply Connected Quasiorthotropic Elastic Domain with Holes and Cracks

Assume that an elastic quasiorthotropic domain S bounded by an external closed contour $L^{(0)}$ contains M holes with the contours $L^{(1)}, L^{(2)}, \dots, L^{(M)}$ and $N-M$ internal curvilinear cracks $L^{(n)}$ ($n = M+1, \dots, N$). Assume that the contours $L^{(n)}$, $n = 0, \dots, N$, are smooth and have no common points (Fig. 3).

Each contour $L^{(n)}$, $n = 0, \dots, N$, is subjected to the action of a self-balanced load:

$$N_n + iT_n = p(t), \quad t \in L^{(n)}, \quad n = \overline{0, M}, \quad (28)$$

$$N_n^+ + iT_n^+ = N_n^- + iT_n^- = p(t), \quad t \in L^{(n)}, \quad n = \overline{M+1, N}.$$

The complex potentials $\Phi_1(z_1)$ and $\Psi_1(z_1)$ are sought in the form (25), where $L_1 = \bigcup L_1^{(k)}$ ($k = \overline{0, N}$). Substituting functions (25) in the boundary conditions (28), we get the following system of singular integral equations:

$$\int_{L_1} [K_1(\tau_1, t_1) g'_1(\tau_1) d\tau_1 + L_1(\tau_1, t_1) \overline{g'_1(\tau_1)} d\bar{\tau}_1] - \frac{1}{2i} \sum_{k=0}^M \frac{\Delta_{kn} M^{(k)}}{(\bar{t}_1 - \bar{z}_k^0)^2} \frac{d\bar{t}_1}{dt_1} + \frac{\delta_n a_n^0}{l} \frac{ds_1}{dt_1} = \pi \tilde{P}_1(t_1), \quad t_1 \in L_1^{(n)}, \quad n = 0, \dots, N, \quad (29)$$

for the $N+1$ unknown functions $g'_1(t_1)$ ($t_1 \in L_1^{(k)}$, $k = 0, 1, \dots, N$). Here, $\tilde{z}_0^0 = z_0^0 = 0 \in S$ (Fig. 3), the kernels $K_1(\tau_1, t_1)$ and $L_1(\tau_1, t_1)$ are determined from relations (18), and the symbols Δ_{kn} and δ_n are given by the formulas

$$\Delta_{kn} = 1 + (\delta_n - 1) \delta_{k0}, \quad \delta_n = \begin{cases} 1, & n = 0, \dots, M, \\ 0, & n = M+1, \dots, N, \end{cases} \quad \delta_{kn} = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$$

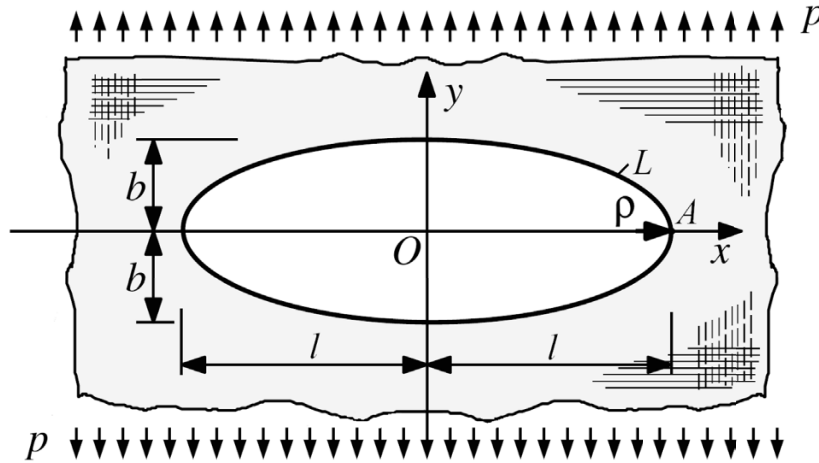


Fig. 4. Elliptic hole in the orthotropic plate subjected to tension along the Oy -axis at infinity.

To the left-hand sides of Eqs. (29), we also add functionals (27) equal to zero if the balance conditions are satisfied. Then the system of integral equations (29) with arbitrary right-hand sides possesses a unique solution under the additional conditions

$$\int_{L_1^{(n)}} g'_1(\tau_1) d\tau_1 = 0, \quad n = M+1, \dots, N,$$

which follow from the condition of uniqueness of displacements in traversing the contours of the cracks.

Setting the parameter $\gamma = 1$ in the relations presented above, we get the well-known results for the isotropic body [15, 19, 20].

Numerical Results

Consider a problem of uniaxial tension of a quasiorthotropic plate weakened by a load-free elliptic hole (plane stressed state, $\gamma = \sqrt{E_x/E_y}$). The axes of the ellipse are parallel to the axes of orthotropy (Fig. 4).

By using the superposition of potentials (13) ($q = \tau = 0$) and (25) ($M = 1$), we arrive at the integral equation (22) and find its numerical solution within the method of quadratures without taking into account the symmetry of the problem [15]. For the sake of comparison, we present the numerical values of the stress intensity factor obtained by using different numbers of the quadrature nodes $\tilde{n}$ (see Table 1) and the analytic expression ($\tilde{n} = \infty$) [21]:

$$k_A = \frac{\sigma_y(l, 0)}{p} = 1 + \frac{2}{\gamma} \sqrt{\frac{l}{\rho}},$$

where $\rho = b^2/l$ is the radius of curvature at the tip of the elliptic hole (Fig. 4). The results presented in Table 1 illustrate the rate of convergence of the numerical solution to the exact solution with the use of the regularizing functional $M^{(1)}$. Substituting the functional $M^{(0)}$ for $M^{(1)}$ in Eq. (22), we get almost the same numerical solution. Although the functionals $M^{(1)}$ and $M^{(0)}$ are equal to zero for all numbers $\tilde{n}$, they guarantee the convergence of the numerical solution for any smooth contour L .

Table 1. Values of the Stress Intensity Factor k_A at the Tip of an Elliptic Hole for the Relative Radius of Curvature $\varepsilon = \rho/l = 1/64$

$\gamma \backslash \tilde{n}$	6	8	10	12	14	16	32	64	∞
$\frac{1}{2}$	39.358	42.725	44.186	43.692	41.924	39.693	31.893	32.901	33
1	16.723	16.350	15.943	15.762	15.800	15.969	16.941	17.000	17
2	7.777	8.195	8.566	8.794	8.909	8.961	9.000	9.000	9

Under conditions of geometric and elastic symmetry about the Ox -axis ($g'_1(\bar{t}_1) = -\overline{g'_1(t_1)}$) or about the Oy -axis ($g'_1(-\bar{t}_1) = -\overline{g'_1(t_1)}$), the efficiency of the numerical method can be significantly increased. In order to take into account the conditions of symmetry about two axes Ox and Oy , it is necessary to somewhat modify Eqs. (22) [15, 19, 20].

CONCLUSIONS

We constructed singular integral equations of the first basic problem of the plane elasticity theory for a quasiorthotropic body in a multiply connected domain containing holes and curvilinear cracks. For the unconditional solvability of these equations, we added certain regularizing functionals equal to zero if the balance conditions are satisfied to the analyzed equations. The efficiency of the numerical solution of the equations is illustrated by analyzing the problem of uniaxial tension of a quasiorthotropic plate with elliptic hole for which it is possible to construct a closed analytic solution.

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