

Stress-Strain State of an Anisotropic Plate with Curved Cracks and Thin Rigid Inclusions

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Abstract—We solve the bending problem for an anisotropic plate with flaws like smooth curved nonoverlapping through cracks and rigid inclusions. The problem is solved by the method of Lekhnitskii complex potentials specified as Cauchy type integrals over the flaw contours with an unknown integrand density function. We use the Sokhotskii–Plemelj formulas to reduce the boundary-value problem to a system of singular integral equations with the additional conditions that the displacements in the plate are single-valued when going around the cut contours and the equilibrium conditions for stress-free rigid inclusions. After the singular integrals are approximated by the Gauss–Chebyshev quadrature formulas, the problem is reduced to solving a system of linear algebraic equations. We study the local stress distribution near flaw tips. We analyze the mutual influence of flaws on the stress distribution character near their vertices and compare the well-known solutions for isotropic plates with the solutions obtained by passing to the limit in the anisotropy parameters (“weakly anisotropic material”) and by using the method proposed here.

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1. INTRODUCTION

The performance of structural members depends on whether they have stress concentrators like cracks and rigid inclusions. In the monograph [1], the Kirchhoff theory and methods of the theory of functions of complex variable were used to systematize the main results about bending problems for isotropic plates with curved cuts or rigid inclusions whose contours have cusps. The solution of the bending problem for elastic plates supported by thin elastic nonclosed rods is reduced to determining the plate–rod contact interaction stresses. These problems lead to integral equations with a special characteristic part [2, 3]. For plates with elastic linear inclusions of variable rigidity, several results can be found in [4, 5]. Determining the unknown contact stresses is reduced here to solving an integro-differential equation with a special characteristic part. A survey of the literature in this field shows that most studies are restricted to isotropic plate materials and simple flaw geometry (rectilinear cracks and inclusions).

The present paper is the first to construct complex potentials for an anisotropic plate in bending with flaws like smooth curved nonoverlapping through cracks and rigid inclusions. We show that the potential representations suggested in [8] and the methods used there to solve bending problems for plates with cracks can be generalized to bending problems for anisotropic plates with rigid curved inclusions. We are the first to obtain a system of singular integral equations and solvability conditions for the problem about the interaction of rigid curved inclusions as well as the interaction of cracks and rigid inclusions. We derive asymptotic formulas for the stress distribution near the tips of a rigid inclusion. We study the local stress distribution near the flaw tips and analyze the mutual influence of flaws on the character of the stress distribution near their tips.

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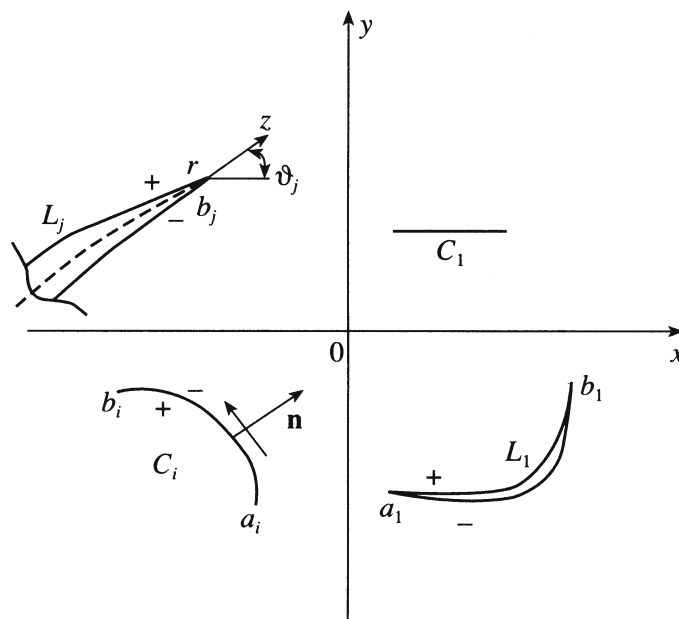


Fig. 1.

2. STATEMENT OF THE PROBLEM

Consider an infinite plate (Fig. 1) made of an anisotropic material and possessing a single elastic symmetry plane parallel to the plate midsurface. The plate is weakened by k curved smooth cuts L_j and m absolutely rigid inclusions C_i . The contours of the cuts and rigid inclusions do not have points of self-intersection. By assumption, the shores of the cuts do not contact with each other and are free from the action of distributed loads:

$$M_{\mathbf{n}}^{\pm} = 0, \quad Q_{\mathbf{n}}^{\pm} = 0. \quad (2.1)$$

In the bending process, the curved rigid inclusions can move as rigid objects and the boundary conditions for them can be represented as [6]

$$\left(\frac{\partial^2 w}{\partial n \partial s} - \frac{1}{\rho} \frac{\partial w}{\partial s} \right)^{\pm} = 0, \quad \left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{\rho} \frac{\partial w}{\partial n} \right)^{\pm} = 0, \quad (2.2)$$

where the plus and minus signs correspond to the left and right shores of the cut or inclusion, w denotes the plate midsurface deflection, and ρ is the current radius of curvature of the rigid inclusion in the midplane of the plate. The differentiation in (2.2) is performed along the contour arc s and along the normal $\mathbf{n}$ to this arc.

The plate is subjected to bending by an arbitrary load, which can be represented by bending and twisting torques at infinity ($M_x^{\infty}, M_y^{\infty}, H_{xy}^{\infty}$) as well as lumped and distributed transverse loads ($\mathbf{P}_z, q(x, y)$). It is required to determine the stress-strain state of the plate.

The stress-strain state of the plate is determined by the method of Lechnitskii complex potentials [7] in which, to solve the problem, it is necessary to construct two functions $F_{\nu}(z_{\nu})$ ($\nu = 1, 2$), analytic in the domain occupied by the plate, of generalized complex coordinates $z_{\nu} = x + \mu_{\nu}y$, so that these functions satisfy the boundary conditions. Here μ_{ν} are the roots of the characteristic equation

$$D_{22}\mu^4 + 4D_{26}\mu^3 + 2(D_{12} + 2D_{66})\mu^2 + 4D_{16}\mu + D_{11} = 0, \quad \mu_1 \neq \mu_2,$$

where D_{ij} are the rigidities of the anisotropic plate; $z = x + iy$ are the complex coordinates of interior points of the plate.

The functions $\Phi_{\nu}(z_{\nu})$ ($\Phi_{\nu}(z_{\nu}) = F_{\nu}''(z_{\nu})$), which give the solution of the problem, can be represented in the form

$$\Phi_{\nu}(z_{\nu}) = \Phi_{\nu}^*(z_{\nu}) + \Phi_{\nu}^{(1)}(z_{\nu}) + \Phi_{\nu}^{(2)}(z_{\nu}), \quad (2.3)$$

where the functions $\Phi_\nu^*(z_\nu)$ determine the stressed state in an infinite plate without cracks and inclusions under the action of an external load (we assume that these functions are known), the functions $\Phi_\nu^{(1)}(z_\nu)$ determine a perturbation of the stress-strain state owing to the presence of curved cuts $L = \bigcup_{j=1}^k L_j$, and $\Phi_\nu^{(2)}(z_\nu)$ determine perturbations of stress-strain state owing to curved rigid inclusions $C = \bigcup_{i=1}^m C_i$. We represent the potentials $\Phi_\nu^{(l)}(z_\nu)$ ($l = 1, 2$) as Cauchy-type integrals with unknown densities $\omega_\nu^{(l)}(\tau)$ ($l = 1, 2$) over the corresponding contours

$$\Phi_\nu^{(1)}(z_\nu) = \frac{1}{2\pi i} \int_L \frac{\omega_\nu^{(1)}(\tau)}{\tau_\nu - z_\nu} d\tau_\nu, \quad \Phi_\nu^{(2)}(z_\nu) = \frac{1}{2\pi i} \int_C \frac{\omega_\nu^{(2)}(\tau)}{\tau_\nu - z_\nu} d\tau_\nu.$$

3. SYSTEM OF INTEGRAL EQUATIONS OF THE PROBLEM

The validity of the boundary conditions (2.1), (2.2) on each of the flaws results in the following relations on the contours L and C :

$$\begin{aligned} a^{(1)}(t)\Phi_1^\pm(t_1) + b^{(1)}(t)\overline{\Phi_1^\pm(t_1)} + \Phi_2^\pm(t_2) &= F^\pm(t), \quad t \in L, \\ a^{(2)}(t)\Phi_1^\pm(t_1) + b^{(2)}(t)\overline{\Phi_1^\pm(t_1)} + \Phi_2^\pm(t_2) &= 0(t), \quad t \in C, \\ \alpha = \frac{q_1\bar{p}_2}{\bar{\mu}_2} - \frac{\bar{q}_2p_1}{\mu_1}, \quad \beta = \frac{\bar{q}_1\bar{p}_2}{\bar{\mu}_2} - \frac{\bar{q}_2\bar{p}_1}{\bar{\mu}_1}, \quad \gamma = \frac{q_2\bar{p}_2}{\bar{\mu}_2} - \frac{\bar{q}_2p_2}{\mu_2}, \quad a^{(1)}(t) &= \frac{\alpha}{\gamma} \frac{M_1(t)}{M_2(t)}, \\ b^{(1)}(t) = \frac{\beta}{\gamma} \frac{\overline{M_1(t)}}{M_2(t)}, \quad a^{(2)}(t) = \frac{M_1(t)(\mu_1 - \bar{\mu}_2)}{M_2(t)(\mu_2 - \bar{\mu}_2)}, \quad b^{(2)}(t) &= \frac{\overline{M_1(t)}(\bar{\mu}_1 - \bar{\mu}_2)}{M_2(t)(\mu_2 - \bar{\mu}_2)}, \\ p_\nu &= D_{11} + D_{12}\mu_\nu^2 + 2D_{16}\mu_\nu, \quad q_\nu = D_{12} + D_{22}\mu_\nu^2 + 2D_{26}\mu_\nu, \\ r_\nu &= D_{16} + D_{26}\mu_\nu^2 + 2D_{66}\mu_\nu, \quad \nu = 1, 2, \\ F^\pm(t) &= \frac{C(t)}{\gamma M_2(t)} \left[\frac{\bar{p}_2}{\bar{\mu}_2} \cos \psi - \bar{q}_2 \sin \psi \right], \quad M_\nu(t) = \mu_\nu \cos \psi - \sin \psi, \end{aligned} \quad (3.1)$$

where $C(t) = C^{(j)}(t)$ ($j = 1, 2, 3, \dots, k$) are the desired unknown real constants on the contours $L = \bigcup_{j=1}^k L_j$; ψ is the angle between the positive normal (directed to the right if the motion starts from the flaw origin a_j to its end b_j) on the contour L (or C) and the x -axis. We use the Sokhotskii–Plemelj formulas, substitute the limit values of the complex potentials (2.3) into the boundary conditions (3.1), and obtain the following system of integral equations for the solution of the problem:

$$\begin{aligned} \int_L \left\{ K_{11}^1(t, \tau) \omega^{(1)}(\tau) + K_{11}^2(t, \tau) \overline{\omega^{(1)}(\tau)} \right\} ds + \int_C \left\{ K_{12}^1(t, \tau) \omega^{(2)}(\tau) + K_{12}^2(t, \tau) \overline{\omega^{(2)}(\tau)} \right\} ds &= F_1^*(t), \\ t \in L, \\ \int_L \left\{ K_{21}^1(t, \tau) \omega^{(1)}(\tau) + K_{21}^2(t, \tau) \overline{\omega^{(1)}(\tau)} \right\} ds + \int_C \left\{ K_{22}^1(t, \tau) \omega^{(2)}(\tau) + K_{22}^2(t, \tau) \overline{\omega^{(2)}(\tau)} \right\} ds &= F_2^*(t), \\ t \in C, \quad \omega^{(l)}(\tau) &= \omega_1^{(l)}(\tau), \quad l = 1, 2. \end{aligned} \quad (3.2)$$

Equations (3.2) should be supplemented with relations between $\omega_1^{(l)}(\tau)$ and $\omega_2^{(l)}(\tau)$ [8]:

$$\omega_2^{(l)}(\tau) = -a^{(l)}(\tau) \omega^{(l)}(\tau) - b^{(l)}(\tau) \overline{\omega^{(l)}(\tau)}, \quad l = 1, 2, \quad \tau = L, C.$$

In Eq. (3.2), the kernels $K_{11}^1(t, \tau)$ and $K_{22}^1(t, \tau)$ contain a Cauchy singularity, the kernels $K_{11}^2(t, \tau)$, $K_{22}^2(t, \tau)$, $K_{12}^1(t, \tau)$, $K_{12}^2(t, \tau)$, $K_{21}^1(t, \tau)$, and $K_{21}^2(t, \tau)$ are functions continuous in T and τ , and the functions F_j^* ($j = 1, 2$) are continuous on L and C .

Equations (3.2) should be supplemented with the single-valuedness conditions for the deflections and tangential displacements when going around the cut contours L_j :

$$\operatorname{Re}\left\{\int_{L_j} \omega^{(1)}(\tau) \left(\tau_1 - \frac{\alpha}{\gamma} \tau_2 - \frac{\bar{\beta}}{\bar{\gamma}} \bar{\tau}_2\right) d\tau_1\right\} = 0, \quad \int_{L_j} \omega^{(1)}(\tau) d\tau_1 = 0. \quad (3.3)$$

For rigid inclusions, the additional conditions are the conditions that the principal moment of the forces acting on each rigid inclusion from the plate is zero (the conditions that the principal vector is zero are satisfied automatically):

$$\int_{C_i} \omega^{(2)}(\tau) d\tau_1 = 0. \quad (3.4)$$

4. NUMERICAL SOLUTION

We introduce parametric representations $\tau^{(i)}(\xi)$ ($i = 1, 2, 3, \dots, k + m$), $\xi \in [-1; +1]$, for the curved open contours L and C and take into account the fact that the integrand densities for the contours L and C can be represented as [3]

$$\omega^{(l)}(\tau(\xi)) = \chi_l(\xi) = \frac{\chi_l^0(\xi)}{\sqrt{1 - \xi^2}} \quad (l = 1, 2).$$

We use the Gauss–Chebyshev quadrature formulas to calculate the singular integrals and the Gauss quadrature formulas to calculate the usual integrals and reduce the system of integral equations (3.2) with the additional conditions (3.3), (3.4) to a system of linear algebraic equations for the values of the unknown integrand density functions at the collocation points on the contours L and C . After this, with any given accuracy, we can find the values of the potentials and hence the stresses at any point of the plate and then calculate the values of the stress intensity coefficients K_1 and K_2 at the tips cracks and rigid inclusions (a neighborhood of the cut tip is shown in the second quadrant in Fig. 1):

$$(M_x, M_y, H_{xy}) = -2 \operatorname{Re} \left[\sum_{\nu=1}^2 (p_\nu, q_\nu, r_\nu) \Phi_\nu(z_\nu) \right],$$

$$M_\vartheta = K_1 (2r)^{-1/2}, \quad H_{n\vartheta} = K_2 (2r)^{-1/2},$$

$$K_1 = -\operatorname{Re} \left\{ \sum_{\nu=1}^2 (p_\nu \sin^2 \vartheta_j + q_\nu \cos^2 \vartheta_j - 2r_\nu \sin \vartheta_j \cos \vartheta_j) [\pm \chi_{\nu j}^0(\mp 1)] \sqrt{\frac{t_{\nu}^{\prime j}(\mp 1)}{\pm (\cos \vartheta_j + \mu_\nu \sin \vartheta_j)}} \right\},$$

$$K_2 = -\operatorname{Re} \left\{ \sum_{\nu=1}^2 [(p_\nu - q_\nu) \cos \vartheta_j \sin \vartheta_j + r_\nu (\cos^2 \vartheta_j - \sin^2 \vartheta_j)] [\pm \chi_{\nu j}^0(\mp 1)] \sqrt{\frac{t_{\nu}^{\prime j}(\mp 1)}{\pm (\cos \vartheta_j + \mu_\nu \sin \vartheta_j)}} \right\},$$

$$t_{\nu}^{\prime j}(\xi) = \frac{dt^j(\xi)}{d\xi}.$$

Here the index j numbers the flaws.

The results of calculations by the method proposed above were compared with the solutions for isotropic plates with flaws. The results for isotropic plates were obtained by passing to the limit in the anisotropy parameters in the numerical solution. Figure 2 presents the dependence of the stress intensity coefficient $K_1(\lambda, \alpha_1)$ for an isotropic plate (material 1 in Table 1) containing two rectilinear cracks

Table 1

N_2	$E_2 \cdot 10^{-4}$ MPa	E_1/E_2	ν_{12}	$G \cdot 10^{-4}$ MPa
1	27.610	1	0.25	11.044
2	1.975	3	0.25	0.863
3	1.104	25	0.25	0.552

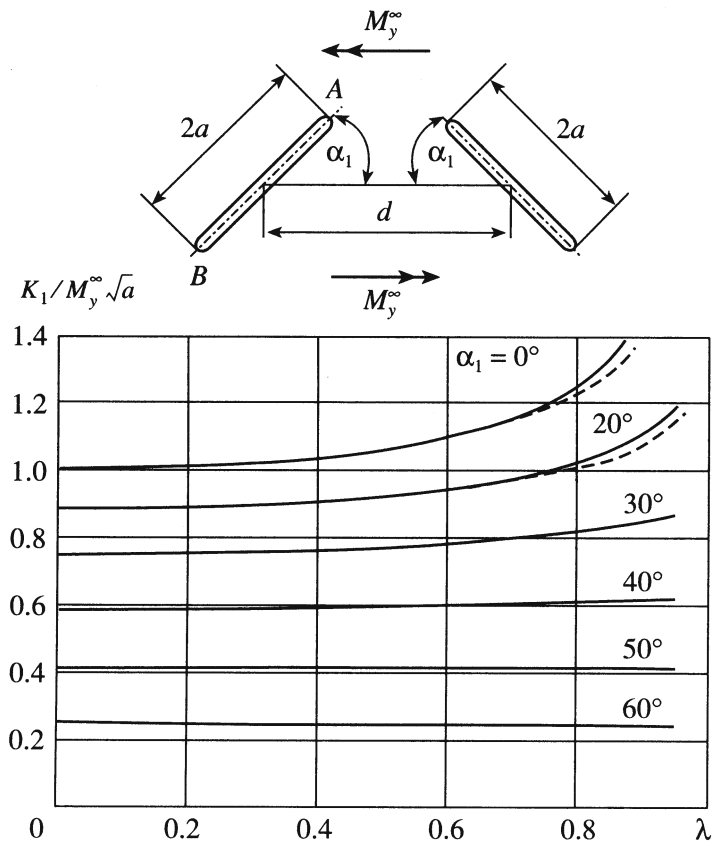


Fig. 2.

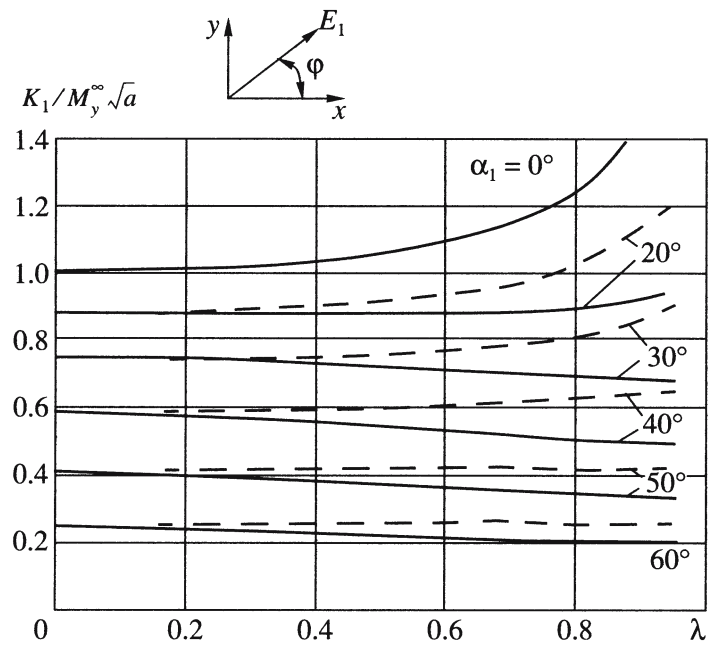


Fig. 3.

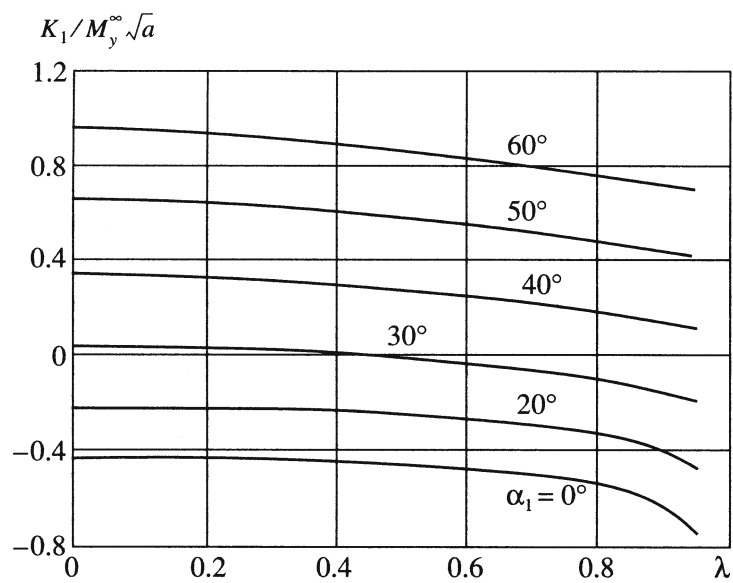


Fig. 4.

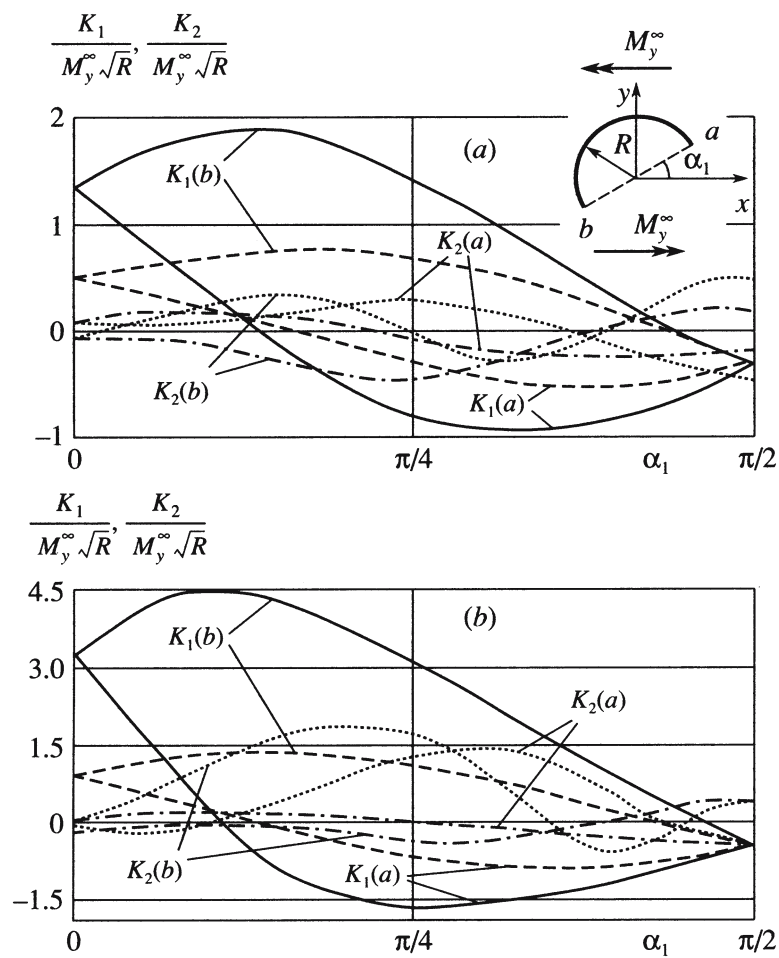


Fig. 5.

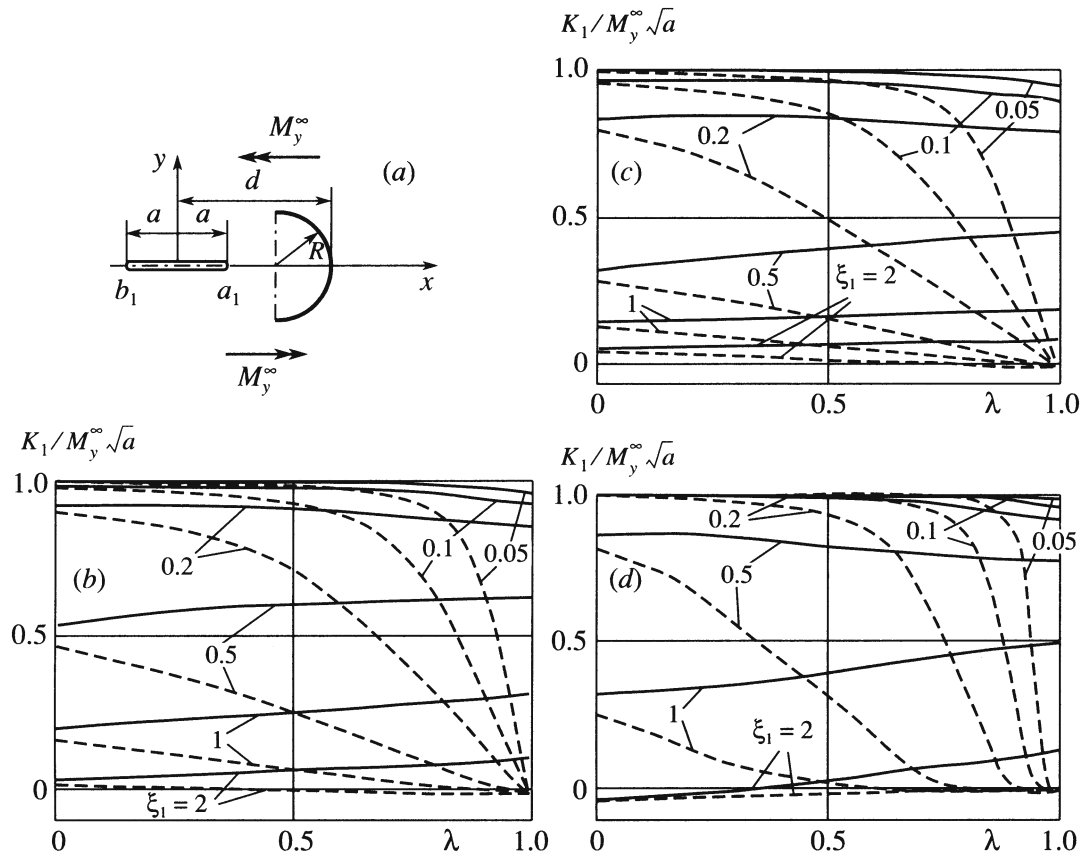


Fig. 6.

Table 2

α	φ	Number of nodes					
		60	40	20	12	4	2
0	0	1.7689	1.7689	1.7688	1.7640	1.5010	1.1457
	$\frac{1}{2}\pi$	1.7689	1.7689	1.7688	1.7640	1.5010	1.1457
$\frac{\pi}{6}$	0	0.6812	0.6812	0.6812	0.6812	0.6583	0.6824
	$\frac{1}{2}\pi$	0.9183	0.9183	0.9183	0.9183	0.8999	0.8145
$\frac{\pi}{3}$	0	0.2	0.2	0.2	0.2	0.1948	0.1841
	$\frac{1}{2}\pi$	0.2503	0.2503	0.2503	0.2503	0.2502	0.2499

and loaded at infinity by uniformly distributed bending moments M_y^∞ . Here the parameter $\lambda = 2a/d$ determines the distance between the cracks, and α_1 is the crack direction angle (α_1 varies from 0° to 60°). The intensity coefficient $K_1(\lambda, \alpha_1)$ for crack tip A was calculated on the areas oriented in the tangential direction at the tip ($\vartheta_j = \alpha_1$). The dotted lines in the figure correspond to the results presented in [9], where the expansion of complex potentials into Laurent series was used to solve this problem.

Figure 3 presents the dependence $K_1(\lambda, \alpha_1)$ for an orthotropic plate (Fig. 2) made of a graphite-epoxy composite (material 3 in Table 1) for various anisotropy angles (φ is the angle between the direction of E_1 and the x -axis). The solid lines correspond to $\varphi = 0$, and the dashed lines correspond to $\varphi = \frac{1}{2}\pi$. The convergence of the computational results can be estimated judging from the data in Table 2, where the dimensionless values of $K_1(\lambda, \alpha_1)$ (for $\lambda = 0.95$) are given for various numbers of collocation nodes ($n = 2, 4, 12, 20, 40$) on the cut contours. It follows from Table 2 that the relative error of the order

of 0.001 in calculations of the stress intensity coefficients is already guaranteed when 20 node points are given on the cut contour, which testifies that the proposed method is highly efficient.

Figure 4 shows the computational results for the stress intensity coefficient $K_1(\lambda, \alpha_1)$ for an isotropic plate containing two rectilinear rigid inclusions (see Fig. 2). It should be noted that, for some inclination angles of the rigid inclusions, compressing stresses appear on the extended surface near the tip of the rigid inclusion.

Figure 5 shows the dependences $K_1(\alpha_1)$ and $K_2(\alpha_1)$ for an unbounded plate that contains a rigid inclusion shaped as a half-circle arc and is bent by moments uniformly distributed at infinity. For glass-epoxy composite (material 2 in Table 1), the computations were performed for the following two cases: $\varphi = 0$, solid and dotted curves; $\varphi = \frac{1}{2}\pi$, dashed and dot-and-dash curves (Fig. 5a). Figure 5b presents the results obtained for the plate made of graphite-epoxy composite ($\varphi = 0$) and an isotropic material (solid, dotted and dashed, dot-and-dash curves, respectively). It follows from the graphs that in the entire range of α_1 the stress intensity coefficient strongly depends on the degree of anisotropy of the plate material.

Figure 6b shows the stress intensity coefficient distribution at the tips a_1 (dashed curves) and b_1 (solid curves) of a rectilinear cut in the plate with a rigid semicircular inclusion (Fig. 6a) depending on $\xi_1 = R/d$ and $\lambda = a/d$ (for the isotropic material). Figures 6c, d show similar graphic dependencies for the plate made of graphite-epoxy composite (the anisotropy angle φ is equal to 0 and $\frac{1}{2}\pi$).

In conclusion, we note that the stress-strain state near the tip of a crack (or a rigid inclusion) is significantly affected by the anisotropy parameters of the plate material, the flaw geometry, the relations between the inclusion and crack dimensions, and the distance between them. The proposed method is realized as a program with a convenient interface, which permits efficiently solving bending problems for anisotropic plates with stress concentrators like crack and rigid inclusions and can be recommended for practical applications.

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