



Stress state of a half-plane with cracks under rigid punch action

V. V. PANASYUK, O. P. DATSYSHYN and H. P. MARCHENKO

Karpenko Physico-Mechanical Institute, National Academy of Sciences of Ukraine, 5 Naukova Str., 79601 Lviv, Ukraine. e-mail: datsyshy@ah.ipm.lviv.ua

Received 24 June 1998; accepted in revised form 30 April 1999

Abstract. A contact problem in elasticity theory for an isotropic half-plane with a set of curvilinear cracks, into which a rigid punch with the foundation of convex shape is indented, is considered. Coulomb friction is assumed to exist between the punch and the half-plane, while the crack faces are under conditions of either stick or smooth contact on contact parts. On the basis of integral representation for Kolosov-Muskhelishvili complex potentials by derivatives of displacement discontinuities along the crack contours and pressure under the punch, the problem is reduced to a system of complex Cauchy type singular integral equations of first and second kind. An algorithm is proposed to find solution of these equations by the method of mechanical quadratures using an iterative procedure. Two numerical examples are presented.

Key words: Curvilinear cracks, punch, friction, smooth contact, stick, singular integral equation.

1. Introduction

It is well known that during the contact of the units of machines and structures, cracks often arise and propagate in the contact zone. These cracks are responsible for the loss of units serviceability. That is why the study of fracture of solids under their contact interaction and, in particular, the problems about the determination of the stress state of solids with cracks initiated by punch pressure are important for engineering practice.

The majority of theoretical investigations were devoted to interaction between a rigid punch and a vertical crack (semi-infinite (Tonoyan and Melkumian, 1970, 1973; Minasian and Tonoyan, 1985), finite internal (Minasian, 1975; Sarkisian et al., 1980; Avetisian et al., 1983) or edge (Minasian and Tonoyan, 1985; Melkumian, 1972; Tonoyan and Minasian, 1975, 1976; Hasebe, 1981; Hasebe et al., 1989; Okumura et al., 1990; Hasebe and Qian, 1995, 1997)) in an elastic isotropic half-plane. An inclined edge crack was considered in (Qian and Hasebe, 1997). In these papers, either individual cases when the crack is open were investigated or contact of the crack faces was neglected. In some papers, a periodic set of vertical cracks (Boyadzhi et al., 1988) and a set of rectilinear arbitrarily oriented cracks (Kudish, 1986) were considered too. As to conditions of interaction between the punch and the half-plane, the cases of stick (Minasian and Tonoyan, 1985; Hasebe, 1981), smooth contact (Tonoyan and Melkumian, 1970, 1973; Minasian, 1975; Sarkisian et al., 1980; Avetisian et al., 1983; Melkumian, 1972; Tonoyan and Minasian, 1975, 1976; Boyadzhi et al., 1988), contact with friction (Hasebe et al., 1989; Okumura et al., 1990; Hasebe and Qian, 1995, 1997; Qian and Hasebe, 1997; Kudish, 1986) were investigated. It should be noted that although the problem of contact of the crack faces in the field of compressive stresses was widely discussed in literature (Savruk, 1981; Hills and Nowell, 1994), it has not been yet investigated if there is a punch.

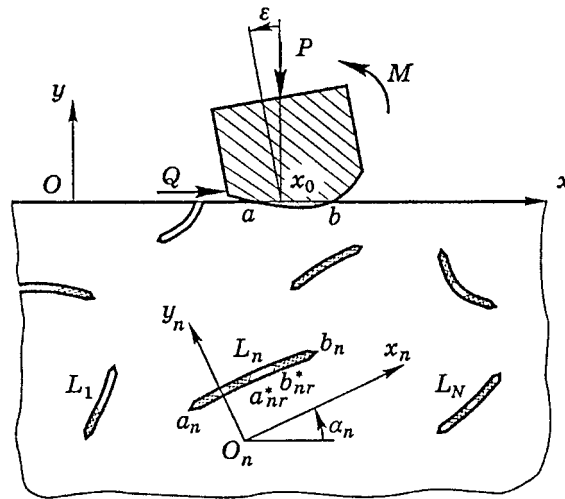


Figure 1. General scheme of the problem.

In the above mentioned papers, investigations were basically carried out by the dual integral equation method (Tonoyan and Melkumian, 1970, 1973; Minasian and Tonoyan, 1985; Minasian, 1975; Melkumian, 1972; Tonoyan and Minasian, 1975, 1976) and the conformal mapping method (Hasebe, 1981; Hasebe et al., 1989; Okumura et al., 1990; Hasebe and Qian, 1995, 1997; Qian and Hasebe, 1997). In the first case, Fredholm integral equations of the second kind were finally obtained. As a result of using the conformal mapping method, an initial problem was reduced to the Riemann-Hilbert problem, the investigations being brought to numerical results. Besides, singular integral equations were obtained for some cases of rectilinear cracks; their solutions were sought as series of Chebyshev polynomials (Sarkisian et al., 1980; Avetisyan et al., 1983) or as power asymptotical series (Kudish, 1986).

The present paper is a continuation of the previous investigations of the authors (Panasyuk et al., 1995, 1996, 1998) and is devoted to solving a contact problem for an elastic isotropic half-plane with a set of curvilinear cracks under the rigid punch action. The singular integral equation method (for details see (Savruk, 1981; Panasyuk et al., 1976, 1977)) which is developed by the authors for two-dimensional contact problems in fracture mechanics is used in the paper. Herewith, Kolosov-Muskhelishvili complex potentials (Muskhelishvili, 1975) are constructed as integral representations with the Cauchy kernels with respect to derivatives of displacement discontinuities along the crack contours and pressure under the punch. As a result, the considered problem is reduced to a system of singular integral equations of first and second kind. Numerical results for the half-plane with a vertical or horizontal crack under the action of the frictionless flat punch are obtained.

2. Problem statement

Let us consider a problem of indentation of a rigid punch with the convex foundation into an elastic isotropic half-plane with a set of N smooth curvilinear cuts (cracks) L_n ($n = 1, \dots, N$). The half-plane is related to the system of coordinates xOy , while the cuts – to the local systems $x_n O_n y_n$ (Figure 1), connected with the basic system xOy by the relation

$z = z_n e^{i\alpha_n} + z_n^0$ ($z = x + iy$, $z_n = x_n + iy_n$, z_n^0 is the affix of the point O_n in the basic system of coordinates, α_n is the angle of the $O_n x_n$ -axis with Ox).

A vertical force P with abscissa of the application point x_0 , a horizontal force Q and a moment M act on the punch. Outside the contact area, $a \leq x \leq b$, the half-plane boundary is unloaded. The punch is considered to be in limiting equilibrium in shear (frictional forces arise between the punch and the half-plane by the Coulomb friction law; $Q = \rho P$, where ρ is the coefficient of friction). Under the action of the external loading, the punch is displaced translationally in the direction parallel to the Oy -axis with simultaneous rotation through a certain small angle ε .

Thus, the boundary conditions of the problem on the half-plane boundary have the form

$$\frac{dv(x)}{dx} = f'(x) + \varepsilon, \quad \tau_{xy}(x) = -\rho\sigma_y(x), \quad a \leq x \leq b, \quad (1)$$

$$\sigma_y(x) \neq i\tau_{xy}(x) = 0, \quad x < a, \quad x > b. \quad (2)$$

We will denote the components of displacement vector on the Ox - and Oy -axes of the basic coordinate system by u and v . The function $f(x)$ describes the punch foundation contour which is a smooth curve.

Under the punch indentation, the crack faces can overlap in compression zones. This causes the emergence of contact stresses. Outside the contact parts the crack faces are unstressed. We will investigate two limiting cases of interaction of the crack faces when face friction is either infinitely large (stick) or absent (smooth contact).

Let L_n^* denotes a set of those parts $a_{nr}^* b_{nr}^*$ ($r = 1, \dots, R_n$) of the n th crack, along which its faces do not contact. Herewith, the parts will be numbered from the beginning an to the end b_n of the crack (see Figure 1), i.e.

$$L_n^* = \bigcup_{r=1}^{R_n} a_{nr}^* b_{nr}^*, \quad n = 1, \dots, N. \quad (3)$$

It should be added that the points a_{nr}^* , b_{nr}^* at $r = 1$ and $r = R_n$ may coincide with the crack tips a_n , b_n . Obviously, for any open crack, $L_n^* = L_n$.

Put down the boundary conditions on the crack faces as follows:

$$N_n^\pm(t_n) + iT_n^\pm(t_n) = 0, \quad t_n \in L_n^*, \quad n = 1, \dots, N, \quad (4)$$

$$u_n^+(t_n) - u_n^-(t_n) + i[v_n^+(t_n) - v_n^-(t_n)] = 0, \quad t_n \in L_n \setminus L_n^*, \quad n = 1, \dots, N, \quad (5)$$

$$N_n^+(t_n) - N_n^-(t_n) + i[T_n^+(t_n) - T_n^-(t_n)] = 0, \quad t_n \in L_n \setminus L_n^*, \quad n = 1, \dots, N, \quad (6)$$

$$T_n^\pm(t_n) = 0, \quad v_n^+(t_n) - v_n^-(t_n) = 0, \quad t_n \in L_n \setminus L_n^*, \quad n = 1, \dots, N, \quad (7)$$

$$N_n^+(t_n) - N_n^-(t_n) = 0, \quad t_n \in L_n \setminus L_n^*, \quad n = 1, \dots, N; \quad (8)$$

the boundary conditions (4)–(6) are valid when stick of the crack faces in contact takes place, while (4), (7), (8) are given for smooth contact of the faces. Here, N_n and T_n are the normal and tangential tractions on the n th crack faces, u_n and v_n the components of displacement vector on the $O_n x_n$ - and $O_n y_n$ -axes of the local coordinate system $x_n O_n y_n$, v_n the normal component of displacements of the crack faces. The superscripts (+) or (–) denote the boundary values

of quantities when approaching the crack contours from the left or from the right. Note that the positive direction of the tangential tractions T_n coincides with that of tracing the region boundary when the region stays all the time on the left (Muskhelishvili, 1975).

Besides, the conditions of punch equilibrium

$$\int_a^b p_0(\zeta) d\zeta = -(P + iQ), \quad (9)$$

$$\int_a^b (\zeta - x_0)p(\zeta) d\zeta = -M \quad (10)$$

must be fulfilled. Here, $p_0(\zeta) = -(1 + i\rho)p(\zeta)$, $p(\zeta)$ is the unknown pressure under the punch.

It should be noted that the borders of the contact parts between the crack faces and also between the punch foundation and the half-plane boundary, generally speaking, are unknown *a priori* and are found from additional equations. The borders of the contact zones on the cracks are found from equations

$$K_{In}(t_{nr}) = 0, \quad t_{nr} = a_{nr}^*, \quad b_{nr}^*, \quad r = 1, \dots, R_n, \quad a_{nr}^* \neq a_n, \quad b_{nr}^* \neq b_n, \quad (11)$$

where K_{In} denotes the Mode I stress intensity factor for the n th crack. The unknown width of the contact area between the punch foundation and the half-plane boundary is determined from the condition that the contact pressure, $p(x) = -\sigma_y(x)$ ($a \leq x \leq b$), at the edge points a and b of this area is equal to zero, i.e.

$$p(a) = 0, \quad p(b) = 0. \quad (12)$$

Obviously, the problem solution has a physical sense if

$$p(x) \geq 0 \quad \text{at} \quad a \leq x \leq b. \quad (13)$$

Further, for convenience, we will use an operator form to write some contour integrals.

Introduce operators $\mathcal{A}\{L_k\}$, $\mathcal{B}\{L_k\}$ according to formulae

$$\begin{aligned} \mathcal{A}\{L_k\}\psi_k(\tau_k) = & \frac{1}{2\pi} \int_{L_k} \left\{ \left[\frac{1}{T_k - z} - \frac{1}{\bar{T}_k - z} \right] e^{i\alpha_k} \psi_k(\tau_k) d\tau_k \right. \\ & \left. + \frac{2i \operatorname{Im} T_k}{(\bar{T}_k - z)^2} e^{-i\alpha_k} \overline{\psi_k(\tau_k)} d\bar{\tau}_k \right\}, \end{aligned} \quad (14)$$

$$\begin{aligned} \mathcal{B}\{L_k\}\psi_k(\tau_k) = & \frac{1}{2\pi} \int_{L_k} \left\{ \left[\frac{1}{(\bar{T}_k - z)^2} - \frac{1}{(T_k - z)^2} \right] \bar{R}_k e^{i\alpha_k} \psi_k(\tau_k) d\tau_k \right. \\ & \left. + \left[\frac{1}{T_k - z} - \frac{1}{\bar{T}_k - z} - 2i \operatorname{Im} T_k \frac{\bar{T}_k + z}{(\bar{T}_k - z)^3} \right] e^{-i\alpha_k} \overline{\psi_k(\tau_k)} d\bar{\tau}_k \right\}, \end{aligned} \quad (15)$$

where

$$T_k = \tau_k e^{i\alpha_k} + z_k^0. \quad (16)$$

The index k indicates that a given quantity is written in the local coordinates. Represent Kolosov-Muskhelishvili complex potentials of the stated problem as function sums

$$\Phi(z) = \Phi_0(z) + \Phi_1(z), \quad \Psi(z) = \Psi_0(z) + \Psi_1(z). \quad (17)$$

The functions $\Phi_0(z)$ and $\Psi_0(z)$ define the stress state of the uncracked half-plane with the boundary loaded by the contact tractions $p_0(x)$. These functions are obtained (Muskhelishvili, 1975) using the boundary condition (2), namely

$$\begin{aligned} \Phi_0(z) &= -\frac{1}{2\pi i} \int_a^b \frac{p_0(\zeta)}{\zeta - z} d\zeta, \\ \Psi_0(z) &= -\frac{1}{2\pi i} \int_a^b \left[\frac{\bar{p}_0(\zeta)}{\zeta - z} - \frac{\zeta p_0(\zeta)}{(\zeta - z)^2} \right] d\zeta. \end{aligned} \quad (18)$$

The functions $\Phi_1(z)$ and $\Psi_1(z)$ describe the half-plane stress state due to displacement discontinuities along the cut contours L_n ($n = 1, \dots, N$) (Savruk, 1981). In view of (14) and (15), these functions are represented as follows:

$$\Phi_1(z) = \sum_{k=1}^N \mathcal{A}\{L_k\} g'_k(\tau_k), \quad \Psi_1(z) = \sum_{k=1}^N \mathcal{B}\{L_k\} g'_k(\tau_k); \quad (19)$$

$$g'_k(\tau_k) = \frac{2G}{i(1 + \varkappa)} \frac{d}{d\tau_k} [u_k^+(\tau_k) - u_k^-(\tau_k) + i(v_k^+(\tau_k) - v_k^-(\tau_k))] \quad (20)$$

with G being the shear modulus, $\varkappa = 3 - 4\mu$ in plane strain and $\varkappa = (3 - \mu)/(1 + \mu)$ in plane stress, μ being the Poisson's ratio.

Construct now singular integral equations of the formulated problem for each case of interaction of the crack faces.

3. Stick of the crack faces in contact

In this case, the boundary conditions of the problem are (1), (2), (4)–(6) and, in accordance with the boundary condition (5), the functions (19) can be rewritten as

$$\Phi_1(z) = \sum_{k=1}^N \mathcal{A}\{L_k^*\} g'_k(\tau_k), \quad \Psi_1(z) = \sum_{k=1}^N \mathcal{B}\{L_k^*\} g'_k(\tau_k). \quad (21)$$

On the basis of representation of the boundary condition for the first basic problem in elasticity theory by Kolosov-Muskhelishvili complex potentials (Muskhelishvili, 1975), we obtain the following formula for the stress distribution along the n th crack faces

$$2 \operatorname{Re} \Phi^\pm(t_n) + \frac{d\bar{t}_n}{dt_n} [t_n \overline{\Phi'^\pm(t_n)} + \overline{\Psi^\pm(t_n)}] = N_n^\pm(t_n) + iT_n^\pm(t_n), \quad t_n \in L_n. \quad (22)$$

By substituting now the complex potentials (17), (18), (21) into the boundary conditions (4) on the crack contours in view of (22), we come to a system of N singular integral equations of the first kind

$$\Omega_n(t_n) - P_n(t_n) = 0, \quad t_n \in L_n^*, \quad n = 1, \dots, N \quad (23)$$

for the functions $g'_n(t_n)$ on the open parts $a_{nr}^* b_{nr}^*$ ($n = 1, \dots, N, r = 1, \dots, R_n$) of the cracks and the pressure $p(x)$. Here,

$$\Omega_n(t_n) = \sum_{k=1}^N \mathcal{D}_n\{L_k^*\} g'_k(\tau_k), \quad (24)$$

$$P_n(t_n) = \int_a^b \left\{ \operatorname{Im} \frac{1 - i\rho}{\zeta - \overline{T'_n}} \frac{d\bar{t}_n}{dt_n} e^{-2i\alpha_n} \right\} p(\zeta) d\zeta, \quad (25)$$

$$T'_n = t_n e^{i\alpha_n} + z_n^0. \quad (26)$$

The operator $\mathcal{D}_n\{L_k\}$ is given by

$$\mathcal{D}_n\{L_k\} \psi_k(\tau_k) = \int_{L_k} [R_{nk}(\tau_k, t_n) \psi_k(\tau_k) d\tau_k + S_{nk}(\tau_k, t_n) \overline{\psi_k(\tau_k)} d\bar{\tau}_k], \quad (27)$$

where, according to (Datsyshyn and Marchenko, 1984),

$$\begin{aligned} R_{nk}(\tau_k, t_n) &= \frac{e^{i\alpha_k}}{2} \left\{ \frac{1}{T_k - T'_n} - \frac{1}{\bar{T}_k - \bar{T}'_n} - 2i \frac{\operatorname{Im} T_k}{(T_k - \overline{T'_n})^2} \right. \\ &\quad \left. - \left[\frac{1}{T_k - \overline{T'_n}} - \frac{1}{\bar{T}_k - \bar{T}'_n} - 2i \operatorname{Im} T_k \frac{T_k + \overline{T'_n} - 2T'_n}{(T_k - \overline{T'_n})^3} \right] \frac{d\bar{t}_n}{dt_n} e^{-2i\alpha_n} \right\}, \\ S_{nk}(\tau_k, t_n) &= \frac{e^{-i\alpha_k}}{2} \left\{ \frac{1}{\bar{T}_k - \bar{T}'_n} - \frac{1}{T_k - T'_n} + 2i \frac{\operatorname{Im} T_k}{(\bar{T}_k - T'_n)^2} \right. \\ &\quad \left. - \left[\frac{T_k - T'_n}{(T_k - \overline{T'_n})^2} + \frac{T_k - T'_n}{(\bar{T}_k - \bar{T}'_n)^2} \right] \frac{d\bar{t}_n}{dt_n} e^{-2i\alpha_n} \right\}. \end{aligned} \quad (28)$$

To obtain a complete system of equations for the stated problem, we use the first of the boundary conditions (1). On the basis of representation of the boundary condition for the second basic problem by Kolosov-Muskhelishvili complex potentials (Muskhelishvili, 1975), we write derivative of complex combination of displacements on the half-plane edge with respect to x as

$$2G \left[\frac{du(x)}{dx} + i \frac{dv(x)}{dx} \right] = \mathfrak{a} \Phi^-(x) - \overline{\Phi^-(x)} - [x \overline{\Phi'^-(x)} + \overline{\Psi^-(x)}]. \quad (29)$$

Proceeding now from the expressions for the potentials (17), (18), (21), we obtain the right-hand part of (29), namely

$$2G \left[\frac{du(x)}{dx} + i \frac{dv(x)}{dx} \right] = \frac{\mathfrak{a} - 1}{2} p_0(x) + \frac{\mathfrak{a} + 1}{\pi} i \left[W(x) + \frac{1}{2} \int_a^b \frac{p_0(\zeta)}{\zeta - x} d\zeta \right], \quad (30)$$

where

$$W(x) = \sum_{k=1}^n \mathcal{U}\{L_k^*\} g'_k(\tau_k) \quad (31)$$

and the operator $\mathcal{U}\{L_k\}$ is defined by

$$\mathcal{U}\{L_k\} \psi_k(\tau_k) = \int_{L_k} \left[\operatorname{Im} \left(\frac{1}{T_k - x} \right) e^{i\alpha_k} \psi_k(\tau_k) d\tau_k + \frac{\operatorname{Im} T_k}{(\bar{T}_k - x)^2} e^{-i\alpha_k} \overline{\psi_k(\tau_k)} d\bar{\tau}_k \right]. \quad (32)$$

Substituting the values of derivatives of displacements (30) into the first of the boundary conditions (1), we come to a singular integral equation of the second kind

$$2 \operatorname{Re} W(x) - \pi \frac{\alpha - 1}{\alpha + 1} \rho p(x) - \int_a^b \frac{p(\zeta)}{\zeta - x} d\zeta = \frac{4\pi G}{\alpha + 1} [f'(x) + \varepsilon], \quad a \leq x \leq b. \quad (33)$$

The solution of the system of the singular integral equations (23), (33) must satisfy the conditions of displacement continuity at the ends of the open parts

$$\int_{a_{nr}^* b_{nr}^*} g_n'(\tau_n) d\tau_n = 0, \quad n = 1, \dots, N, \quad r = 1, \dots, R_n \quad (34)$$

and the conditions of punch equilibrium (9), (10).

Thus, the singular integral equations (23), (33) and conditions (9), (10), (34) allows to find the unknown functions $g_n'(t_n)$ ($n = 1, \dots, N$), $p(x)$ and also the quantity ε .

On the basis of the obtained solution using (6), (22), one can also obtain a relation to determine contact stresses on the contact parts of the crack faces as

$$N_n(t_n) + iT_n(t_n) = \frac{1}{\pi} [\Omega_n(t_n) - P_n(t_n)], \quad t_n \in L_n \setminus L_n^*. \quad (35)$$

The solution of the system of the constructed singular integral equations allows to calculate the stress intensity factors (SIF) at the crack tips and at the points where the crack faces begin to contact by

$$K_{In}(t_{nr}) - iK_{II n}(t_{nr}) = \mp \lim_{t_n \rightarrow t_{nr}} [\sqrt{2\pi|t_n - t_{nr}|} g_n'(t_n)], \quad (36)$$

$$t_{nr} = a_{nr}^*, b_{nr}^*, \quad n = 1, \dots, N, \quad r = 1, \dots, R_n;$$

the upper sign (−) conforms to the points a_{nr}^* , while the lower sign (+) – to the points b_{nr}^* . Note that for open cracks (when $a_{nr}^* = a_n$, $b_{nr}^* = b_n$), formula (36) coincides with the known one (Savruk, 1981).

4. Smooth contact of the crack faces

In this case, the boundary conditions of the problem are of the form (1), (2), (4), (7), (8). Following Savruk (1981), we represent the unknown functions $g_n'(t_n)$ ($n = 1, \dots, N$) in (17) as sums of two functions, namely

$$g_n'(t_n) = g_{1n}'(t_n) + g_{2n}'(t_n). \quad (37)$$

These functions are expressed by discontinuity of the normal v_v and tangential v_s components of displacement vector to the cut contour L_n by

$$g_{1n}(t_n) = \frac{2G}{1 + \alpha} [v_v^+(t_n) - v_v^-(t_n)] \frac{dt_n}{ds_n}, \quad (38)$$

$$g_{2n}(t_n) = -\frac{2iG}{1 + \alpha} [v_s^+(t_n) - v_s^-(t_n)] \frac{dt_n}{ds_n}, \quad (39)$$

where s_n is the arc abscissa of a point t_n on the contour L_n . Then, in view of the second group of the boundary conditions (7), the complex potentials $\Phi_1(z)$, $\Psi(z)$ are finally written as

$$\begin{aligned}\Phi_1(z) &= \sum_{k=1}^N [\mathcal{A}\{L_k^*\}g'_{1k}(\tau_k) + \mathcal{A}\{L_k\}g'_{2k}(\tau_k)], \\ \Psi_1(z) &= \sum_{k=1}^N [\mathcal{B}\{L_k^*\}g'_{1k}(\tau_k) + \mathcal{B}\{L_k\}g'_{2k}(\tau_k)].\end{aligned}\quad (40)$$

By substituting the complex potentials (17), (18), (40) into the boundary conditions (4), (7) in view of (22), we obtain the following system of $2N$ singular integral equations

$$\operatorname{Re}[\Omega_n(t_n) - P_n(t_n)] = 0, \quad t_n \in L_n^*, \quad n = 1, \dots, N, \quad (41)$$

$$\operatorname{Im}[\Omega_n(t_n) - P_n(t_n)] = 0, \quad t_n \in L_n, \quad n = 1, \dots, N \quad (42)$$

to determine the functions $g'_{1n}(t_n)$ along the open parts $a_{nr}^*b_{nr}^*$ ($n = 1, \dots, N, r = 1, \dots, R_n$) of the cracks, the functions $g'_{2n}(t_n)$ along the contours L_n ($n = 1, \dots, N$) and the pressure $p(x)$. Here,

$$\Omega_n(t_n) = \sum_{k=1}^N [\mathcal{D}_n\{L_k^*\}g'_{1k}(\tau_k) + \mathcal{D}_n\{L_k\}g'_{2k}(\tau_k)] \quad (43)$$

and the quantity $P_n(t_n)$ is determined by (25). To complete the system of equations (41), (42), it is necessary to add equations

$$\operatorname{Im} \left[g_{1n}(t_n) \frac{d\bar{t}_n}{ds_n} \right] = 0, \quad t_n \in L_n^*, \quad n = 1, \dots, N, \quad (44)$$

$$\operatorname{Re} \left[g_{2n}(t_n) \frac{d\bar{t}_n}{ds_n} \right] = 0, \quad t_n \in L_n, \quad n = 1, \dots, N, \quad (45)$$

which follow immediately from (38), (39) and also equation (33), where

$$W(x) = \sum_{k=1}^N [\mathcal{U}\{L_k^*\}g'_{1k}(\tau_k) + \mathcal{U}\{L_k\}g'_{2k}(\tau_k)]. \quad (46)$$

We obtain the expression for the function $W(x)$ by analogy with the previous case on the basis of the first of the boundary condition (1) using (17), (18), (40).

Besides, the additional conditions

$$\int_{a_{nr}^*b_{nr}^*} g'_{1n}(\tau_n) d\tau_n = 0, \quad n = 1, \dots, N, \quad r = 1, \dots, R_n, \quad (47)$$

$$\int_{L_n} g'_{2n}(\tau_n) d\tau_n = 0, \quad n = 1, \dots, N \quad (48)$$

and (9), (10) must be satisfied.

By solving the obtained equations in view of (8), (22), we take a relation

$$N(t_n) = \frac{1}{\pi} \operatorname{Re} [\Omega_n(t_n) - P_n(t_n)], \quad t_n \in L_n \setminus L_n^* \quad (49)$$

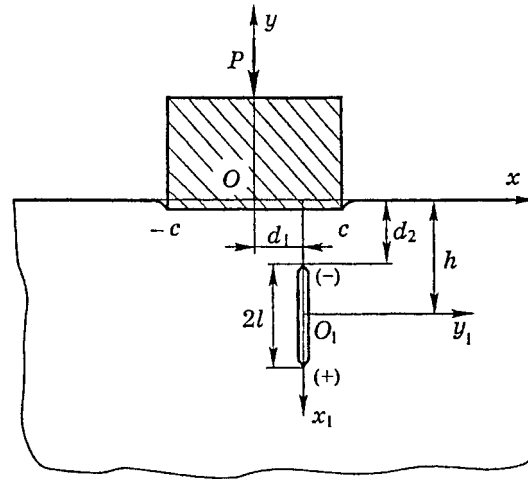


Figure 2. A half-plane with a vertical crack under punch action.

to determine normal stresses on the contact parts of the crack faces.

To calculate the SIF, we use formula (36). However, unlike the preceding case, the factors Π_n are sought only at the crack tips ($a_{nr}^* = a_n$, $b_{nr}^* = b_n$).

It should be noted that for the case of stick between the punch foundation and the half-plane boundary, singular integral equations have been constructed by the authors previously (Panasyuk et al., 1995, 1996).

5. An algorithm for solving the singular integral equations of the problem

Numerical solution for the constructed singular integral equations can be obtained by the method of mechanical quadratures with the use of Gauss-Jacoby integration formulae (Erdogan et al., 1973). However, since the contact zones of the crack faces are unknown *a priori* at problem statement, an iterative procedure is used for their finding. The algorithm consists in the following. As an initial approximation of the contact zones we assume the parts where normal stresses along the crack lines in the uncracked half-plane under the punch action are compressive. Further, on the basis of the numerical solutions of the suitable integral equations, we calculate values of the factors K_{ln} by (36) at the points which are initial approximations for the points a_{nr}^* , b_{nr}^* . These factors may be different from zero. Then, by holding a small increment of lengths of the parts found at the preceding stage, we obtain the first approximations of the contact zones of the crack faces. It allows to calculate new values of the factors K_{ln} at the points which are the first approximations for the points a_{nr}^* , b_{nr}^* . Approximation process goes on until the factors K_{ln} become close to zero with a required accuracy (in accordance with condition (11)). An accuracy of the obtained results can be checked by formulae (35), (49), taking into account that normal stresses on the contacting parts of the crack faces should be negative.

A similar approach is applied when the contact zone between the punch foundation and the half-plane boundary is unknown. To find the points a and b , we have relations (12) which define unambiguously these points if we take condition (13) into account too. Herewith, both the contact parts between the crack faces and the contact zone between the punch foundation and the half-plane boundary are searched simultaneously.

6. Numerical examples

6.1. A HALF-PLANE WITH A VERTICAL CRACK UNDER PUNCH ACTION

Let a frictionless flat punch be indented by a force P into an elastic half-plane weakened by a near-surface internal vertical crack of length $2l$. The half-plane is related to the system of coordinates xOy , whose the Oy -axis coincides with the punch axis and the line of the action of the force P , while the crack – to the local system of coordinates $x_1O_1y_1$, as shown in Figure 2. Assume that the punch does not rotate under loading (i.e., $\varepsilon = 0$).

Using the algorithm proposed in the previous section, it has been established that for all the positions of the vertical crack near the punch foundation, the crack faces contact throughout its length. Obviously, under stick of its faces, the crack is no longer a stress concentrator.

Investigate now the case of smooth contact of the crack faces. Proceeding from (24), (32), (33), (41)–(46), we finally obtain the following singular integral equations

$$\int_{-l}^l R(\tau, t) \varphi(\tau) d\tau + \int_{-c}^c (\zeta, t) p(\zeta) d\zeta = 0, \quad |t| < l, \quad (50)$$

$$\int_{-l}^l V(\tau, t) \varphi(\tau) d\tau - \int_{-c}^c \frac{p(\zeta)}{\zeta - x} d\zeta = 0, \quad |x| \leq c \quad (51)$$

with the kernels

$$R(\tau, t) = \frac{1}{\tau - t} + \frac{1}{w(\tau, t)} + 2 \frac{\tau + h}{w^2(\tau, t)} - 4 \frac{(\tau + h)^2}{w^3(\tau, t)}, \quad w(\tau, t) = \tau + t + 2h, \\ U(\zeta, t) = -\operatorname{Im} \frac{t+h}{(\zeta - T')^2}, \quad T' = d_1 - i(t + h), \quad (52)$$

$$V(\tau, x) = 2 \left[\operatorname{Im} \frac{1}{T-x} - \operatorname{Re} \frac{\tau+h}{(\bar{T}-x)^2} \right], \quad T = d_1 - i(\tau + h)$$

to determine the functions

$$\varphi(t) = i g_2(t) = -\frac{2G}{1+\kappa} \frac{d}{dt} [u^+(t) - u^-(t)], \quad |t| < l \quad (53)$$

and $p(x)$ ($|x| \leq c$). Here and further, the indices denoting the ordinal crack number are omitted.

The solution of the system equations (50), (51) must satisfy the additional conditions

$$\int_{-l}^l \varphi(\tau) d\tau = 0, \quad (54)$$

$$\int_{-c}^c p(\zeta) d\zeta = P, \quad (55)$$

$$\int_{-c}^c \zeta p(\zeta) d\zeta = -M. \quad (56)$$

Note that condition (56) serves to determine the moment M which restrains the punch from rotation.

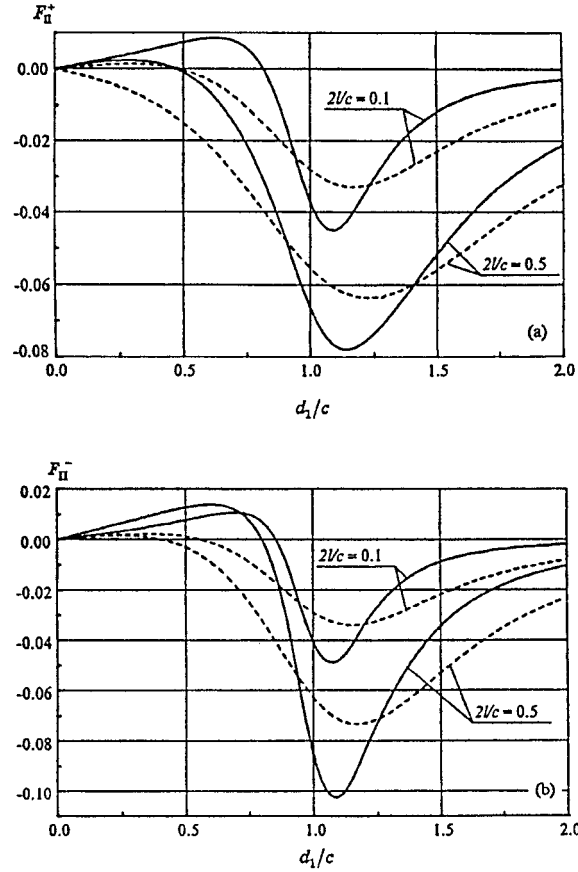


Figure 3. Normalized Mode II stress intensity factors at the tips of a vertical crack, $F_{II}^{\pm} = K_{II}^{\pm} \sqrt{c} / (P \sqrt{\pi})$, vs. d_1/c for $d_2/c = 0.2, 0.5$ and $2l/c = 0.1, 0.5$; solid lines are for $d_2/c = 0.2$ while dashed lines are for $d_2/c = 0.5$. (a) F_{II}^+ for the right tip, (b) F_{II}^- for the left tip.

In Appendix, a numerical technique is presented to solve the system of the singular integral equations (50), (51) by the method of mechanical quadratures and also final formulae are given to determine the SIF at the crack tips and the pressure under the punch.

Plots of the variation of the normalized Mode II SIF, $F_{II}^{\pm} = K_{II}^{\pm} \sqrt{c} / (P \sqrt{\pi})$, depending on relative distance d_1/c of the crack from the punch axis for different crack lengths and distances from the half-plane boundary are shown in Figure 3. Numerical data analysis evidences that the Mode II stress intensity is maximum when the crack approaches the punch edge ($d_1/c \approx 1.1$ for $2l/c = 0.1$ and $d_1/c \approx 1.2$ for $2l/c = 0.5$). When the crack is placed along the Oy -axis ($d_1 = 0$), by virtue of problem symmetry, the factors K_{II} are equal to zero at both the crack tips. Then, the crack is no longer a stress concentrator and does not influence the pressure distribution under the punch.

In Table 1, the numerical values of the pressure $p(x)$ under the punch for the most dangerous crack positions are give when the SIF in magnitude are maximum. It is seen that in comparison with the uncracked half-plane, the pressure near the punch edge where the crack lies is lower and, on the contrary, near the other edge is higher. Besides, the crack presence has an insignificant effect on the value and the character of the pressure distribution under the punch; the pressure value changes not more than by 7 percent.

Table 1. Pressure under a frictionless flat punch which is indented to a half-plane with a vertical crack.

$2l/c$	d_1/c	d_2/c	$cp(x)/P$				
			$x/c = -0.9$	$x/c = -0.5$	$x/c = 0$	$x/c = 0.5$	$x/c = 0.9$
0	—	—	0.73025	0.36755	0.31831	0.36755	0.73025
0.1	1.1	0.2	0.7257	0.3679	0.3188	0.3685	0.7307
0.1	1.2	0.5	0.7292	0.3677	0.3185	0.3675	0.7305
0.5	1.1	0.2	0.6780	0.3732	0.3254	0.3709	0.7385
0.5	1.2	0.5	0.7194	0.3707	0.3206	0.3636	0.7354

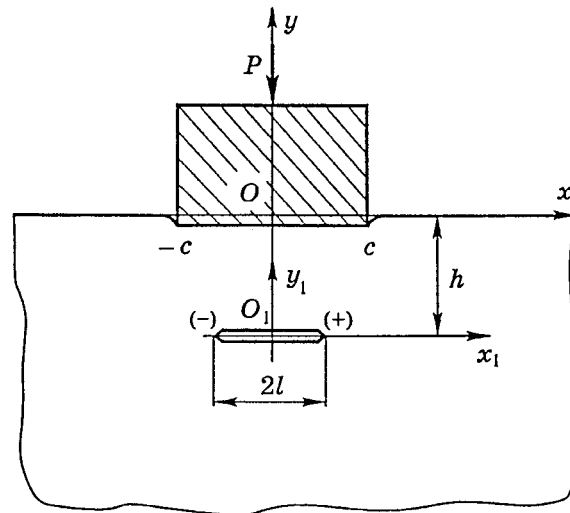


Figure 4. A half-plane with a horizontal crack under punch action.

It should be noted that the particular case for symmetric position of the crack ($d_1 = 0$) was considered in (Sarkisian et al., 1980). However, the results obtained there turned out to be erroneous since they evidenced about the lack of contacting the crack faces.

6.2. A HALF-PLANE WITH A HORIZONTAL CRACK UNDER PUNCH ACTION

The case of indentation of a frictionless flat punch into an elastic half-plane with a horizontal crack of length $2l$, placed symmetrically about the punch axis, is also considered (Figure 4). As in the previous case, it has been established that the crack faces contact throughout its length.

Under smooth contact of the crack faces, the problem is reduced to solving the singular integral equations (50), (51) with the kernels

$$R(\tau, t) = \frac{1}{\tau - t} + \operatorname{Re} \left[w^{-1}(\tau, t) - \frac{4h^2}{w^3(\tau, t)} \right] + \operatorname{Im} \frac{2h}{w^2(\tau, t)}, \quad w(\tau, t) = t - \tau + 2ih,$$

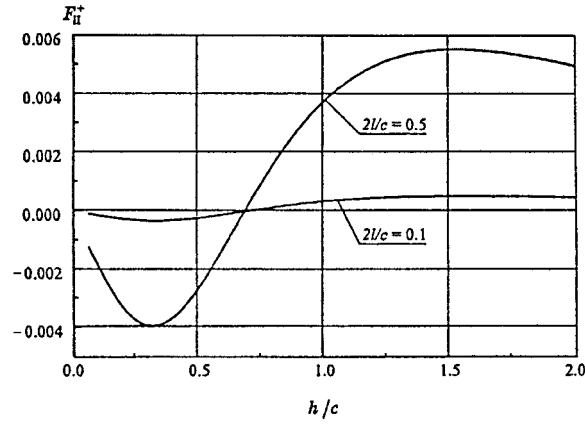


Figure 5. Normalized Mode II stress intensity factors at the right tip of a horizontal crack, $F_{II}^+ = K_{II}^+ \sqrt{c} / (P \sqrt{\pi})$, vs. h/c for $2l/c = 0.1, 0.5$.

Table 2. Pressure under a frictionless flat punch which is indented into a half-plane with a horizontal crack.

$2l/c$	h/c	$cp(x)/P$		
		$x/c = 0$	$x/c = 0.5$	$x/c = 0.9$
0	—	0.31831	0.36755	0.73025
0.1	0.3	0.31832	0.36755	0.73025
0.1	1.5	0.31831	0.36755	0.73025
0.5	0.3	0.32112	0.36713	0.73029
0.5	1.5	0.31825	0.36754	0.73031

$$U(\zeta, t) = \text{Im} \frac{h}{(\zeta - t + ih)^2}, \quad (57)$$

$$V(\tau, x) = -2\text{Im} \frac{h}{(\tau + ih - x)^2},$$

and the additional conditions (54), (55) remain unchanged.

The dependence of the normalized Mode II SIF, $F_{II}^+ = K_{II}^+ \sqrt{c} / (P \sqrt{\pi})$ ($F_{II}^- = -F_{II}^+$), on relative distance of the crack from the half-plane boundary is shown in Figure 5. It is seen that in moving the crack away from the boundary, the factor K_{II}^+ reverses the sign and takes two extremum values at $h/c \approx 0.3$ and $h/c \approx 1.5$.

As seen in Table 2, the crack presence in fact does not influence the pressure distribution under the punch.

7. Conclusions

Applications of the singular integral equation method to two-dimensional contact problems in linear elastic fracture mechanics have been presented.

A problem for an elastic isotropic half-plane with a set of curvilinear cracks, which is under the action of a rigid punch with the foundation of convex shape, has been considered. Frictional forces are assumed to act between the punch and the half-plane by the Coulomb friction law, while the crack faces are under conditions of either stick or smooth contact on the contact parts. The problem has been reduced to solving a system of complex Cauchy type singular integral equations of first and second kind. An algorithm which is based on using the method of mechanical quadratures and an iterative procedure has been proposed to find solution of these equations.

The presented approach allows to consider various configurations of cracks and the punch foundation and also general conditions of interaction between the crack faces and between the punch and the half-plane. Besides, this approach allows to find effective numerical solution of obtained singular integral equations.

Numerical analysis has been made for the particular cases of indentation of a frictionless punch with the flat horizontal foundation into a half-plane weakened by a near-surface internal vertical crack or a horizontal crack placed symmetrically about the punch axis. It has been established that for various positions of the considered cracks, they are fully closed.

Under smooth contact of the crack faces:

- (1) The Mode II stress intensity at the tips of the vertical crack is maximum when the crack lies near the punch edge. The crack which is placed along the punch axis is no longer a stress concentrator.
- (2) The Mode II stress intensity at the tips of the horizontal crack is maximum when the crack lies not near the punch foundation but at some distance from the half -plane boundary ($h/c \approx 1.5$; h is the crack depth, $2c$ is the punch foundation length).
- (3) The presence of the vertical or horizontal crack near the punch foundation has an insignificant effect on the pressure distribution under the punch.

Under stick of the crack faces, the vertical crack or the horizontal one do not influence the half-plane stress state.

Appendix

We seek numerical solution of the system of the singular integral equations (50), (51) in the following way. First, by changing the variables

$$\begin{aligned} \tau &= l\xi, \quad \zeta = c\xi, \quad |\xi| \leq 1, \\ t &= l\eta, \quad x = c\eta, \quad |\eta| < 1, \end{aligned} \tag{A1}$$

we reduce the system and the additional conditions (54)–(56) to a normalized form

$$\int_{-1}^1 [R(\xi, \eta)\varphi(\xi) + U(\xi, \eta)p(\xi)] d\xi = 0, \quad |\eta| < 1, \tag{A2}$$

$$\int_{-1}^1 \left[V(\xi, \eta)\varphi(\xi) - \frac{p(\xi)}{c(\xi - \eta)} \right] d\xi = 0, \quad |\eta| < 1, \tag{A3}$$

$$\int_{-1}^1 \varphi(\xi) d\xi = 0, \tag{A4}$$

$$\int_{-1}^1 p(\xi) d\xi = P, \quad (\text{A5})$$

$$c \int_{-1}^1 \xi p(\xi) d\xi = -M. \quad (\text{A6})$$

Here, for convenience, the symbols for functions are retained unchanged, i.e.

$$\begin{aligned} R(\xi, \eta) &= R(l\xi, l\eta), \quad U(\xi, \eta) = U(l\xi, l\eta), \quad V(\xi, \eta) = V(l\xi, l\eta), \\ \varphi(\xi) &= l\varphi(l\xi), \quad p(\xi) = cp(c\xi). \end{aligned} \quad (\text{A7})$$

Further, by extracting in the functions $\varphi(\eta)$, $p(\eta)$ the root singularities at the points $\eta = -1$ and $\eta = 1$, we obtain representations

$$\varphi(\eta) = \frac{u(\eta)}{\sqrt{1-\eta^2}}, \quad p(\eta) = \frac{q(\eta)}{\sqrt{1-\eta^2}}; \quad (\text{A8})$$

the functions $u(\eta)$ and $q(\eta)$ are continuous in the interval $-1 \leq \eta \leq 1$. Applying now Gaussian quadratures to the integral equations (A2)–(A6) in view of equalities (A8), we come to a system of $2N + 1$ linear algebraic equations for $2n + 1$ unknowns $u(\xi_k)$, $q(\xi_k)$ ($k = 1, \dots, N$) and M

$$\sum_{k=1}^N [R(\xi_k, \eta_m)u(\xi_k) + U(\xi_k, \eta_m)q(\xi_k)] = 0, \quad (m = 1, \dots, N-1), \quad (\text{A9})$$

$$\sum_{k=1}^N \left[V(\xi_k, \eta_m)u(\xi_k) - \frac{q(\xi_k)}{c(\xi_k - \eta_m)} \right] = 0, \quad (m = 1, \dots, N-1), \quad (\text{A10})$$

$$\sum_{k=1}^N u(\xi_k) = 0, \quad (\text{A11})$$

$$\pi \sum_{k=1}^N q(\xi_k) = NP, \quad (\text{A12})$$

$$\pi c \sum_{k=1}^N \xi_k q(\xi_k) = -NM. \quad (\text{A13})$$

Here, ξ_k and η_m are the zeros of the Chebyshev polynomials of first, $T_N(\xi) = \cos(N \cos^{-1} \xi)$, and second, $U_{N-1}(\eta) = \sin(N \cos^{-1} \eta) / \sqrt{1-\eta^2}$, kind, respectively.

On the basis of solution of the system of equations (A9)–(A12), using representations of the sought functions by the Lagrange interpolation polynomial in Chebyshev's points ξ_k as (Savruk, 1981)

$$\psi(\xi) = -\frac{1}{N} \sum_{k=1}^N (-1)^k \frac{T_N(\xi) \sqrt{1-\xi_k^2}}{\xi - \xi_k} \psi(\xi_k), \quad (\text{A14})$$

one can find the stress intensity factors at the crack tips (see formula (36))

$$K_{II}^{\pm} = \pm \sqrt{\pi/l} u(\pm 1) \quad (\text{A15})$$

and the pressure in the contact area

$$p(\eta) = -\frac{T_N(\eta)}{N\sqrt{1-\eta^2}} \sum_{k=1}^N (-1)^k \frac{\sqrt{1-\xi_k^2}}{\eta-\xi_k} q(\xi_k), \quad |\eta| < 1. \quad (\text{A16})$$

In (A15), the upper sign (+) concerns the right crack tip, while the lower sign (−) – the left tip, and the function $u(\xi)$ at the points $\xi = +1, -1$ takes the values

$$u(1) = -\frac{1}{N} \sum_{k=1}^N (-1)^k \cot\left(\frac{2k-1}{4N}\pi\right) u(\xi_k), \quad (\text{A17})$$

$$u(-1) = \frac{1}{N} \sum_{k=1}^N (-1)^{k+N} \tan\left(\frac{2k-1}{4N}\pi\right) u(\xi_k). \quad (\text{A18})$$

Acknowledgements

The authors are grateful to Professor M.P. Savruk for useful discussions and comments on this work. This work was partly supported by the State Fund of Fundamental Researches of the Ministry for Science and Technology of Ukraine, under Grant No. 1.4/174.

References

- Avetisyan, Ye.P., Khachatryan, O.Kh. and Shaginian, S.S. (1983). On a mixed boundary value problem in elasticity theory for a half-plane weakened by an internal rectilinear cut. *Uchenyie Zapiski Yerevanskogo Universiteta, Yestestvennye Nauki* **1**, 28–33.
- Boyadzhi, A.H., Buryshkin, M.L. and Radiollo, M. V. (1988). On action of a rigid punch on a half-plane weakened by a regular set of cracks. *Prikladnaya Matematika i Mekhanika* **52**, 856–860.
- Datsyshyn, O.P. and Marchenko, H.P. (1984). Interaction between curvilinear cracks and the boundary of an elastic half-plane. *Physicochemical Mechanics of Materials* **20** (5), 64–71.
- Erdogan, F., Gupta, G.D. and Cook, T.S. (1973). The numerical solutions of singular integral equations. *Methods of Analysis and Solutions of Crack Problems* (Edited by G.S. Sih), Noordhoff International Publishing, Leyden, 368–425.
- Hasebe, N. (1981). An edge crack in a semi-infinite plate welded to a rigid stiffener. *Proceedings of Japan Civil Engineers* **314**, 149–157.
- Hasebe, N., Okumura, M. and Nakamura, T. (1989). Frictional punch and crack in plane elasticity. *Journal of Engineers Mechanics* **115**, 1137–1149.
- Hasebe, N. and Qian, J. (1995). Circular inclined punch problem with two corners to contact with a half-plane with a surface crack. *Proceedings of the 2nd International Conference on Computational Methods in Contact Mechanics*, Computational Mechanics Publications, Southampton, Boston, 159–166.
- Hasebe, N. and Qian, J. (1997). Circular rigid punch with one smooth and another sharp ends on a half-plane with edge crack. *Journal of Applied Mechanics* **64**, 73–79.
- Hills, D.A. and Nowell, D. (1994). *Mechanics of Fretting Fatigue*, Kluwer Academic Publishers, Dordrecht.
- Kudish, I.I. (1986). Contact problem in elasticity theory for bodies with cracks. *Prikladnaya Matematika i Mekhanika* **50**, 1020–1033.
- Melkumian, S.A. (1972). Contact problem for a half-plane with a vertical finite cut. *Doklady Akademii Nauk Arnlanskoj SSR* **55**, 87–93.

- Minasian, A.F. (1975). Asymmetric contact problem for a half-plane with a vertical cut. *Doklady Akademii Nauk Armianskoy SSR* **61**, 82–89.
- Minasian, A.F. and Tonoyan, V.S. (1985). Contact problem for a half-plane with a vertical cut in stick. *Mechanics of Deformable Bodies and Structures*, Akademiya Nauk Armianskoy SSR, Yerevan, 286–297.
- Muskhelishvili, N.I. (1975). *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff International Publishing, Leiden.
- Okumura, M., Hasebe, N. and Nakamura, T. (1990). Crack due to wedge-shaped punch with friction. *Journal of Engineers Mechanics* **116**, 2173–2185.
- Panasyuk, V.V., Datsyshyn, O.P. and Marchenko, H.P. (1995). Contact problem for a half-plane with cracks under action of a rigid punch on its boundary. *Physicochemical Mechanics of Materials* **31** (6), 7–16.
- Panasyuk, V.V., Datsyshyn, O.P. and Marchenko, H.P. (1996). On punch pressure on a half-plane with cracks. *Proceedings of the 9th Conference on Strength and Plasticity*, Vol. 1, Profservis, Moscow, 150–155.
- Panasyuk, V.V., Datsyshyn, O.P. and Marchenko, H.P. (1998). An internal crack under a punch in an elastic solids. *Fracture Mechanics and Physics of Construction Materials and Structures* (Edited by O.Ye. Andreikiv and J.J. Luchko), Issue 3, Kameniar, Lviv, 170–177.
- Panasyuk, V.V., Savruk, M.P. and Datsyshyn, O.P. (1976). *Stress Distribution Around Cracks in Plates and Shells*, Naukova Dumka, Kyiv.
- Panasyuk, V.V., Savruk, M.P. and Datsyshyn, O.P. (1977). A general method of solution of two-dimensional problems in the theory of cracks. *Engineering Fracture Mechanics* **9**, 481–497.
- Qian, J. and Hasebe, N. (1997). Circular rigid punch on a semi-infinite plane with an oblique edge crack subjected to concentrated forces or point dislocations. *Proceedings of Japan Civil Engineers* **570**, 149–159.
- Sarkisian, V.S., Hrihorian, E.Kh. and Shaginian, S.S. (1980). On two boundary value problems in elasticity theory for a half-plane weakened by an internal rectilinear cut. *Uchenyie Zapiski Yerevanskogo Universiteta, Yestestvennye Nauki* **2**, 49–58.
- Savruk, M.P. (1981). *Two-Dimensional Problems of Elasticity for Bodies with Cracks*, Naukova Dumka, Kyiv.
- Tonoyan, V.S. and Melkumian, S.A. (1970). Contact problem for a half-plane with a vertical cut. *Doklady Akademii Nauk Annianskoy SSR* **51**, 144–149.
- Tonoyan, V.S. and Melkumian, S.A. (1973). On symmetric indentation of two rigid similar punches into an elastic half-plane with a vertical semi-infinite cut. *Doklady Akademii Nauk Annianskoy SSR* **57**, 282–288.
- Tonoyan, V.S. and Minasian, A.F. (1975). Asymmetric contact problem for a half-plane with a vertical finite cut. *Doklady Akademii Nauk Annianskoy SSR* **61**, 289–297.
- Tonoyan, V.S. and Minasian, A.F. (1976). On symmetric indentation of two rigid similar punches into an elastic half-plane with a vertical finite cut. *Doklady Akademii Nauk Annianskoy SSR* **62**, 36–42.