

Two-Dimensional Thermal Stresses and Displacements in an Arbitrarily Inhomogeneous Elastic Layer

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Keywords: arbitrarily inhomogeneous layer, plane problem, thermoelasticity, FGM

Abstract. This paper presents an analytical approach for solution of the plane-strain problem on elastic equilibrium of a layer whose material properties are arbitrary functions of the transversal coordinate. The layer is stressed by distributed temperature field under given displacement of its limiting surfaces. By making use of the explicit solution of the relevant problem in terms of stresses, the boundary tractions are determined by the given boundary displacements and temperature field on the basis of established one-to-one relations. In such manner, the original problem is reduced to the problem with boundary conditions in terms of stresses.

Introduction

The problem on control and optimization of mechanical performance of inhomogeneous solids necessitates establishment of one-to-one relations between the stresses and displacements, which are valid in all points of the body, including its boundary. Besides the prominent advantages in solution of the direct problems on thermomechanics of inhomogeneous solids with boundary conditions in terms of displacements or mixed ones (both stresses and displacements are given on the boundary), these relations allow for avoiding the physically incorrect results [1, 2], as well as for development of vital integral conditions of compatibility for the strain-tensor components. For their establishment, one needs to have an explicit analytical solution of the relevant problem, which expresses the requested-for stresses or displacements through the given boundary conditions of either type. Obtaining of such solutions to the problems of thermomechanics for arbitrarily inhomogeneous solids presents a challenge due to the fact that the governing equations of these problems are with unknown variable coefficients, which makes it impossible, in general, to solve the problems by means of classical techniques. To overcome this difficulty, the method of integral equations can be employed [3].

In this paper, the one-to-one relations between the stresses and displacements on the limiting surfaces of an arbitrarily crosswise-inhomogeneous layer are established for the case of plane-strain problem. On the basis of correct integration of the Cauchy relations, the displacements are found in terms of strains, meanwhile the latter ones are expressed through the stresses, including the boundary, by making use of the explicit solution of the foregoing problem in terms of stresses [4].

Formulation of the Problem

Consider elastic equilibrium of an infinite layer $\{|x| < \infty, |y| \leq 1, |z| < \infty\}$ under the plane strain condition ($e_z = 0$). Here x, y , and z are the dimensionless Cartesian coordinates and e_z is the normal strain in z -direction. Assume the material properties of the layer to be arbitrary functions of the transversal coordinate y . In the absence of body forces, the arising plane problem is governed by the equilibrium equations

$$\partial \sigma_x / \partial x + \partial \sigma_{xy} / \partial y = 0, \quad \partial \sigma_{xy} / \partial x + \partial \sigma_y / \partial y = 0 \quad (1)$$

and compatibility equation in terms of strains

$$\partial^2 e_x / \partial y^2 + \partial^2 e_y / \partial x^2 = \partial^2 e_{xy} / \partial x \partial y. \quad (2)$$

Here $\sigma_x, \sigma_y, \sigma_{xy}$ and e_x, e_y, e_{xy} are the stress- and strain-tensor components, respectively. The strains are related to the stresses as

$$2Ge_x = (1-\nu)\sigma - \sigma_y + \alpha ET, \quad 2Ge_y = \sigma_y - \nu\sigma + \alpha ET, \quad Ge_{xy} = \sigma_{xy}, \quad (3)$$

where $\nu = \nu(y)$ denotes the Poisson ratio, $E = E(y)$ and $G = G(y)$ are the Young and shearing moduli, respectively, $\alpha = \alpha(y)$ stands for the linear thermal expansion coefficient, $T = T(x, y)$ is given temperature field, and $\sigma = \sigma_x + \sigma_y$ is the in-plane total stress.

Assume the layer to be locally disturbed by the temperature field T (the stresses and displacements vanish at the points of infinity). We also assume that the displacement of the boundaries $y = \pm 1$ are known and read

$$u(x, 1) = u_1(x), \quad u(x, -1) = u_2(x), \quad v(x, 1) = v_1(x), \quad v(x, -1) = v_2(x). \quad (4)$$

Here u and v are the longitudinal and transversal displacements, respectively, which are related to the strains by the Cauchy equations

$$e_x = \partial u / \partial x, \quad e_y = \partial v / \partial y, \quad e_{xy} = \partial u / \partial y + \partial v / \partial x. \quad (5)$$

By solving equations (1) – (3), (5), we intend to determine the stress-tensor and displacement-vector components under the boundary conditions (4) and given non-uniform temperature field for arbitrary variation of the elastic properties in (3) with coordinate y .

Solution Construction

Due to the distribution of temperature and boundary conditions (4), the normal and shearing tractions

$$\sigma_y(x, 1) = -p_1(x), \quad \sigma_y(x, -1) = p_2(x), \quad \sigma_{xy}(x, 1) = q_1(x), \quad \sigma_{xy}(x, -1) = q_2(x) \quad (6)$$

arise on the boundaries $y = \pm 1$. We can manage to determine these unknown tractions under the given temperature field and boundary displacements (4). Then the solution of the formulated problem (1) – (5) can be easily found by making use of the technique [3, 4].

By taking the boundary conditions (4) into account, the first and second relations of (5) can be solved for the displacements as

$$2u(x, y) = \int_{-\infty}^{\infty} e_x(\xi, y) \operatorname{sgn}(x - \xi) d\xi, \quad 2v(x, y) = v_1 + v_2 + \int_{-1}^1 e_y(x, \eta) \operatorname{sgn}(y - \eta) d\eta. \quad (7)$$

By letting $x \rightarrow \pm\infty$ in the first equation of (7) and $y = \pm 1$ in the second one and taking conditions (4) into account, we arrive at the integral conditions

$$\int_{-\infty}^{\infty} e_x(x, y) dx = 0, \quad \int_{-1}^1 e_y(x, y) dy = v_1 - v_2. \quad (8)$$

Substitution of (7) into the third equation of (5) yields the formula

$$2e_{xy} = \frac{d}{dx}(v_1 + v_2) + \int_{-\infty}^{\infty} \frac{\partial e_x(\xi, y)}{\partial y} \operatorname{sgn}(x - \xi) d\xi + \int_{-1}^1 \frac{\partial e_y(x, \eta)}{\partial x} \operatorname{sgn}(y - \eta) d\eta, \quad (9)$$

which presents the integro-differential compatibility condition for the strain-tensor components. Note that equation (9) is equivalent to equation (2) under the following fitting conditions

$$\frac{dv_1}{dx} = e_{xy}(x, 1) - \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial e_x(\xi, 1)}{\partial y} \operatorname{sgn}(x - \xi) d\xi, \quad \frac{dv_2}{dx} = e_{xy}(x, -1) - \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial e_x(\xi, -1)}{\partial y} \operatorname{sgn}(x - \xi) d\xi. \quad (10)$$

To treat the obtained equations, we apply the Fourier integral transformation

$$\bar{f}(y) = \int_{-\infty}^{\infty} f(x, y) \exp(-isx) dx, \quad (11)$$

where s is the transformation parameter and $i^2 = -1$. Application of transformation (11) to conditions (10) and the first formula in (8) yields

$$\bar{e}_x(1) = is\bar{u}_1, \quad \bar{e}_x(-1) = is\bar{u}_2, \quad \bar{e}_{xy}(1) = -is^{-1} d\bar{e}_x(1)/dy + is\bar{v}_1, \quad \bar{e}_{xy}(-1) = -is^{-1} d\bar{e}_x(-1)/dy + is\bar{v}_2. \quad (12)$$

By making use of the first and third formulae of (3), conditions (12) can be given as

$$\begin{aligned} (1 - v_j)\bar{\sigma}_j + \bar{p}_j &= 2G_j(is\bar{u}_j - \alpha_j(1 + v_j)\bar{T}_j), \\ \frac{d}{dy}\left(\frac{1 - v_j}{G_j}\bar{\sigma}_j\right) + \bar{p}_j \frac{d}{dy}\left(\frac{1}{G_j}\right) - \frac{is\bar{q}_j}{G_j} &= 2s^2\bar{v}_j - 2\frac{d}{dy}(\alpha_j(1 + v_j)\bar{T}_j). \end{aligned} \quad (13)$$

Here $j = 1, 2$ and “1” stand for those functions with argument $y = 1$, meanwhile “2” stands for $y = -1$. By putting the explicit solution [4] of the problem (1) – (3), (6) into relations (13), we arrive at the one-to-one relations between the stresses and displacements on the boundary of inhomogeneous layer. From these relations, we can find the boundary tractions in terms of the boundary displacements, and vice versa. Together with solution [4], this allows us to solve the problems with boundary conditions in terms of stresses, displacements, or ones of mixed type.

Numerical Results and Discussion

Consider computation of boundary tractions (6) on the sides $y = \pm 1$, which are rigidly fixed ($u_1 = u_2 = v_1 = v_2 = 0$). The material properties are assumed to be in the following form: $E = E_0$, $v = v_0$, $\alpha = \alpha_0(1 + y^m)$, where the subscript “0” denotes the constant values. Note that by letting $m = 0$, we obtain the case of homogeneous material. The layer is stressed by supplying to its sides $y = \pm 1$ the temperature $T(x, \pm 1) = \tau_0 \exp(-x^2)$. The corresponding temperature field can be found by solving a classical heat-transport equation with constant heat conductivity.

In Fig. 1, the boundary tractions are shown for $m = 0$ (homogeneous material), $m = 1$ (linear variation of the thermal expansion coefficient), and $m = 3$ (cubical polynomial). The normal tractions are even functions of the longitudinal coordinate x , meanwhile the shearing tractions are odd functions. As we can observe, $p_1 = p_2$ and $q_1 = -q_2$ in the case of homogenous material and the problem appears to be symmetrical about $y = 0$. If we allow the thermal expansion coefficient to vary

with the transversal coordinate y (the cases $m = 1$ and $m = 3$), then the tractions on sides $y = -1$ and $y = 1$ are different. In the latter case, the normal and shearing tractions on the side $y = 1$ are greater by the magnitudes than ones on $y = -1$. This can be explained by lesser thermal expansion of the side $y = -1$ and, thus, lower thermal stresses due to its clamping.

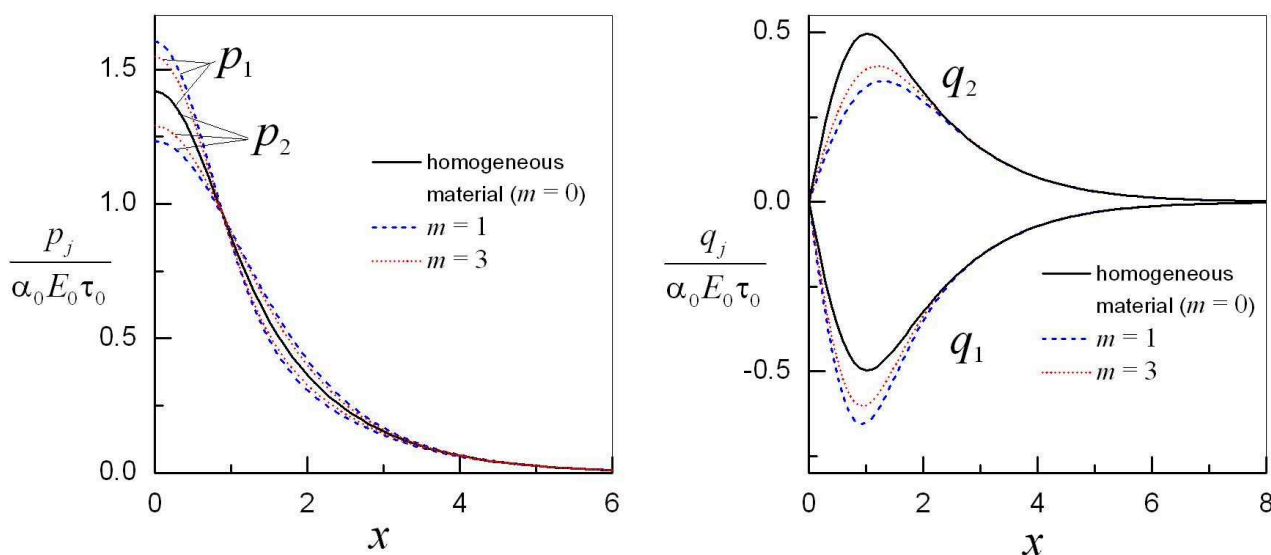


Fig.1 Normal and shearing boundary tractions in a heated layer with clamped sides

Conclusions

By making use of the one-to-one relations set up between the tractions and the displacements on the boundary of an inhomogeneous layer, the quasi-static boundary value problem of the plane thermoelasticity with boundary conditions in terms of displacement is reduced to solution of the problem when the tractions are prescribed on the boundary. Establishment of the relations between tractions and displacements on the boundary allows for solution of the relevant problems with all type boundary conditions, i.e., in terms of stresses, displacement, or the mixed ones.

Acknowledgments

The 1st and 3rd authors gratefully acknowledge particular financial support of this research by National Science Council (ROC) under grant NSC 102-2811-E-002-018. The 2nd author acknowledges the support by the joint grant of Ukrainian National Academy of Sciences and Russian Foundation of Basic Research (No 0114U005082).

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10.4028/www.scientific.net/AMM.627.141

DOI References

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