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October 10–13
in Nara, Japan

Editors Sei Ueda, Yasutomo Uetsuji

**Proceedings of the Fourth Asian Conference on
Mechanics of Functional Materials and Structures**

October 10-13, 2014, Nara, Japan

ACMFMS 2014

Edited by

**Sei Ueda and Yasutomo Uetsuji
Osaka Institute of Technology**

**2014 by Organizing Committee, ACMFMS2014,
Department of Mechanical Engineering,
Faculty of Engineering,
Osaka Institute of Technology,
5-16-1 Omiya, Asahi-ku, Osaka, 535-8585 Japan**

ISBN 978-4-9904352-1-9

Elastic stresses and displacements in anisotropic nonhomogeneous solids

Yu. Lozynskyy¹, Yu. Tokovyy² and C. -C. Ma³

^{1,2} Pidstryhach Institute for Applied Problems of Mechanics and Mathematics,
National Academy of Sciences of Ukraine, 3-B Naukova St., 79060 Lviv, Ukraine

^{2,3} Department of Mechanical Engineering, National Taiwan University,
No 1 Roosevelt Rd. Sec. 4, 10617 Taipei, Taiwan ROC

An analytical approach for solution of the two-dimensional thermoelasticity problem for an infinitely long orthotropic strip, whose material moduli vary arbitrarily with its width, is presented when the boundary conditions are imposed in terms of stresses, displacements or both of them (mixed boundary conditions). On the basis of correct integration of the Cauchy relations, the displacements are univocally expressed in terms of strains and the vital integral compatibility conditions for the latter ones are formulated. By making use of the explicit solution of the relevant problem in terms of stresses along with foregoing expressions, one-to-one relations between the stresses and displacements on the boundary are obtained.

Keywords: *nonhomogeneous solids, one-to-one stress-displacement relations, explicit solution*

1 Introduction

Construction of correct solutions to the problems of elasticity and thermoelasticity with different-type boundary conditions (i.e., in terms of either stresses, displacements, or mixed ones: when both stresses and displacements are given on the boundary) calls for establishment of the fundamental one-to-one relations between the stresses and displacements which are valid for all points of a deformable solid including its boundary. These relations allow for bringing the constructed solution in the agreement with fundamental principles of the continua mechanics [1] as well as for treatment of all possible cases of boundary conditions in a similar manner. To obtain such relations, one needs an explicit solution to the direct problem for either type of the boundary conditions. Obtaining of such solutions to the problems of thermomechanics for arbitrarily nonhomogeneous solids presents a challenge due to the fact that the governing equations of these problems are with unknown variable coefficients, which makes it impossible, in general, to solve the problems by means of classical techniques. To overcome this difficulty, the method of integral equations can be employed [2], which, along with the method of resolvent-kernel solution, allows for constructing the solutions to the elasticity and thermoelasticity problems in terms of stresses for arbitrarily nonhomogeneous solids.

In this paper, we present an approach for solution of a plane problem of thermoelasticity for an infinitely long orthotropic strip with elastic moduli arbitrarily varying from one its side to another under boundary conditions imposed for displacements, stresses or both of them. Establishment of the relations between the

boundary stresses and displacements, which have been achieved by correct integration of the Cauchy relations, allowed for reduction of different types of boundary conditions to the one case, when the tractions are given on the boundary. Solution of the problem for the latter case has been obtained in our previous works for arbitrary dependence of the material properties on the transversal coordinate [3].

2 Formulation of the Problem

We consider a plane thermoelasticity problem for an orthotropic strip $D = \{(x, y) \in (-\infty, +\infty) \times [-b, b]\}$, whose elastic properties are arbitrary functions of coordinate y . Here (x, y) and b are the dimensionless Cartesian coordinates and a dimensionless parameter, respectively. In the absence of body forces, the problem is governed by the equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \quad (1)$$

the compatibility equation

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \partial y}, \quad (2)$$

the Cauchy equations

$$e_x = \frac{\partial u}{\partial x}, \quad e_y = \frac{\partial v}{\partial y}, \quad e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad (3)$$

² tokovyy@gmail.com

and the physical relations

$$\begin{aligned} e_x &= a_{11}\sigma + a_{12}\sigma_y + \alpha_1 T, \\ e_y &= a_{21}\sigma + a_{22}\sigma_y + \alpha_2 T, \\ e_{xy} &= g\sigma_{xy}. \end{aligned} \quad (4)$$

Here σ_i, σ_{xy} and e_i, e_{xy} denote the stress- and strain-tensor components, respectively, $i = \{x, y\}$, u and v are elastic displacements,

$$\begin{aligned} a_{11} &= \tilde{a}_{11}/E_x, \quad a_{12} = -\tilde{a}_{12}/E_y - a_{11} = -\tilde{a}_{21}/E_x - a_{11}, \\ a_{22} &= \tilde{a}_{22}/E_y + \tilde{a}_{12}/E_y = \tilde{a}_{22}/E_y + \tilde{a}_{21}/E_x, \\ a_{21} &= -\tilde{a}_{12}/E_y = -\tilde{a}_{21}/E_x, \quad g = 1/G_{xy}, \\ \tilde{a}_{11} &= \begin{cases} 1, \\ 1 - \nu_{zx}\nu_{xz}, \end{cases} \quad \tilde{a}_{22} = \begin{cases} 1, \\ 1 - \nu_{zy}\nu_{yz}, \end{cases} \quad \tilde{a}_{12} = \begin{cases} \nu_{xz}, \\ \nu_{yz} + \nu_{zx}\nu_{yz}, \end{cases} \\ \tilde{a}_{21} &= \begin{cases} \nu_{zy}, \\ \nu_{zy} + \nu_{xz}\nu_{yz}, \end{cases} \quad \alpha_1 = \begin{cases} \alpha_x, \\ \alpha_x + \alpha_z\nu_{zx}, \end{cases} \quad \alpha_2 = \begin{cases} \alpha_y, \\ \alpha_y + \alpha_z\nu_{zy}, \end{cases} \end{aligned}$$

$E_x(y)$ and $E_y(y)$ are the Young moduli in x - and y -directions, respectively, $\nu_{ij}(y)$ denotes the Poisson ratio describing the contraction in i -direction by extension in j -direction ($i, j = x, y, z$), $G_{xy}(y)$ is the shearing modulus in xOy -plane; $\alpha_i(y)$ are the linear thermal expansion coefficients in i -direction ($i = \{x, y, z\}$), $T(x, y)$ is the given temperature field, $\sigma = \sigma_x + \sigma_y$, and the upper lines under the left-braces in the foregoing expressions correspond to the plane-stress state, meanwhile the second one corresponds to the plane-strain state.

We intend to solve the set of equations (1) – (4) under the following boundary conditions:

$$\begin{aligned} u(x, b) &= u_1(x), \quad u(x, -b) = u_2(x), \\ v(x, b) &= v_1(x), \quad v(x, -b) = v_2(x) \end{aligned} \quad (5)$$

imposed for the displacements at $y = \pm b$. We also assume that the thermal disturbance is local and thus the temperature field, stresses and displacements vanish at points of infinity $|x| \rightarrow \infty$.

3 Solution Technique

By taking the boundary conditions (5) into account, the first and second relations of (3) can be solved for the displacements as

$$\begin{aligned} 2u(x, y) &= \int_{-\infty}^{\infty} e_x(\xi, y) \operatorname{sgn}(x - \xi) d\xi, \\ 2v(x, y) &= v_1 + v_2 + \int_{-b}^b e_y(x, \eta) \operatorname{sgn}(y - \eta) d\eta. \end{aligned} \quad (6)$$

By letting $x \rightarrow \pm\infty$ in the first equation of (6) and $y = \pm 1$ in the second one and taking conditions (5) into account, we arrive at the integral conditions

$$\int_{-\infty}^{\infty} e_x(x, y) dx = 0, \quad \int_{-b}^b e_y(x, y) dy = v_1 - v_2. \quad (7)$$

Putting (6) into the third equation (3) yields

$$\begin{aligned} 2e_{xy} &= \frac{d}{dx}(v_1 + v_2) + \int_{-\infty}^{\infty} \frac{\partial e_x(\xi, y)}{\partial y} \operatorname{sgn}(x - \xi) d\xi \\ &\quad + \int_{-b}^b \frac{\partial e_y(x, \eta)}{\partial x} \operatorname{sgn}(y - \eta) d\eta. \end{aligned} \quad (8)$$

Equation (8) presents the integro-differential compatibility condition for the strain-tensor components. Note that equation (8) is equivalent to equation (2) under the following fitting conditions:

$$\begin{aligned} \frac{dv_1}{dx} &= e_{xy}(x, b) - \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial e_x(\xi, b)}{\partial y} \operatorname{sgn}(x - \xi) d\xi, \\ \frac{dv_2}{dx} &= e_{xy}(x, -b) - \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial e_x(\xi, -b)}{\partial y} \operatorname{sgn}(x - \xi) d\xi. \end{aligned} \quad (9)$$

By applying the Fourier integral transform to equations (7) and (9), we can obtain the following relations

$$\begin{aligned} \bar{e}_x(b) &= is\bar{u}_1, \quad \bar{e}_{xy}(b) = -\frac{i}{s} \frac{d\bar{e}_x(b)}{dy} + is\bar{v}_1, \\ \bar{e}_x(-b) &= is\bar{u}_2, \quad \bar{e}_{xy}(-b) = -\frac{i}{s} \frac{d\bar{e}_x(-b)}{dy} + is\bar{v}_2. \end{aligned} \quad (10)$$

Here s is the transformation parameter, $i^2 = -1$ and the overline denotes the transformation of a corresponding function.

Note that the distribution of temperature and boundary conditions (5) induce the normal and shearing tractions

$$\begin{aligned} \sigma_x(x, b) &= -p_1(x), \quad \sigma_x(x, -b) = p_2(x), \\ \sigma_{xy}(x, b) &= q_1(x), \quad \sigma_{xy}(x, -b) = q_2(x) \end{aligned} \quad (11)$$

on the sides $y = \pm b$. We can manage to determine the unknown tractions (11) by making use of the solution to the problem (1) – (4), (11) in terms of stresses [3], which has the form

$$\begin{aligned} \bar{\sigma}(y) &= \bar{p}_1\chi_1(y) + \bar{p}_2\chi_2(y) + \bar{q}_1\chi_3(y) \\ &\quad + \bar{q}_2\chi_4(y) + \chi_r(y), \end{aligned} \quad (12)$$

where

$$\chi_1(y) = [(I_{12} \operatorname{sh} sb - I_{22} \operatorname{ch} sb) f_A(y) + (I_{21} \operatorname{ch} sb - I_{11} \operatorname{sh} sb) f_B(y)] / (s\gamma),$$

$$\chi_2(y) = \frac{1}{s} \left(f_p(y) + \int_{-b}^b f_p(\xi) \mu(y, \xi) d\xi \right) + [(I_{12} \operatorname{sh} sb + I_{22} \operatorname{ch} sb) f_A(y) - (I_{11} \operatorname{sh} sb + I_{21} \operatorname{ch} sb) f_B(y)] / (s\gamma),$$

$$\chi_3(y) = i[(I_{22} \operatorname{sh} sb - I_{12} \operatorname{ch} sb) f_A(y) + (I_{11} \operatorname{ch} sb - I_{21} \operatorname{sh} sb) f_B(y)] / (s\gamma),$$

$$\chi_4(y) = \frac{i}{s} \left(f_q(y) + \int_{-b}^b f_q(\xi) \mu(y, \xi) d\xi \right) + i[(I_{22} \operatorname{sh} sb + I_{12} \operatorname{ch} sb) f_A(y) - (I_{21} \operatorname{sh} sb + I_{11} \operatorname{ch} sb) f_B(y)] / (s\gamma),$$

$$\chi_T(y) = f_T(y) + \int_{-b}^b f_T(\xi) \mu(y, \xi) d\xi,$$

$$\left\{ \begin{matrix} f_A(y) \\ f_B(y) \end{matrix} \right\} = \frac{1}{a_{11}(y)} \left\{ \begin{matrix} \operatorname{ch} sy \\ \operatorname{sh} sy \end{matrix} \right\} + \int_{-b}^y \left\{ \begin{matrix} \operatorname{ch} s\xi \\ \operatorname{sh} s\xi \end{matrix} \right\} \frac{R(y, \xi)}{a_{11}(\xi)} d\xi,$$

$$\left\{ \begin{matrix} f_p(y) \\ f_q(y) \\ f_T(y) \end{matrix} \right\} = \left\{ \begin{matrix} p(y) \\ q(y) \\ \Theta(y) \end{matrix} \right\} + \int_{-b}^y \left\{ \begin{matrix} p(\xi) \\ q(\xi) \\ \Theta(\xi) \end{matrix} \right\} R(y, \xi) d\xi,$$

$$p(y) = \frac{1}{a_{11}} \int_{-b}^y \left[\left(\frac{d^2 \beta_1(\eta)}{d\eta^2} + s^2 \beta_2(\eta) \right) \operatorname{sh} s(\eta - y) \operatorname{ch} s(\eta + b) + s^2 (2\beta_1(\eta) - g(\eta)) \operatorname{ch} s(y - \eta) \operatorname{sh} s(\eta + b) \right] d\eta,$$

$$q(y) = \frac{1}{a_{11}} \left[s(2\beta_1(-b) - g(-b)) \operatorname{sh} s(y + b) - \int_{-b}^y \left(\frac{d^2 \beta_1(\eta)}{d\eta^2} + s^2 \beta_2(\eta) \right) \operatorname{sh} s(y - \eta) \operatorname{sh} s(\eta + b) + s^2 (2\beta_1(\eta) - g(\eta)) \operatorname{ch} s(y - \eta) \operatorname{ch} s(\eta + b) \right] d\eta,$$

$$\Theta(y) = \frac{-s}{a_{11}} \left[\frac{\alpha_1 \bar{T}}{s} + \int_{-b}^y (\alpha_1(\xi) - \alpha_2(\xi)) \bar{T}(\xi) \operatorname{sh} s(y - \xi) d\xi \right],$$

$$\left\{ \begin{matrix} I_{11} \\ I_{12} \end{matrix} \right\} = \int_{-b}^b \left\{ \begin{matrix} f_A(y) \\ f_B(y) \end{matrix} \right\} \operatorname{sh} sy dy, \quad \left\{ \begin{matrix} I_{21} \\ I_{22} \end{matrix} \right\} = \int_{-b}^b \left\{ \begin{matrix} f_A(y) \\ f_B(y) \end{matrix} \right\} \operatorname{ch} sy dy,$$

$$\gamma = I_{11} I_{22} - I_{12} I_{21}, \quad \beta_1 = \frac{\tilde{a}_{11} + \tilde{a}_{21}}{E_x}, \quad \beta_2 = \frac{\tilde{a}_{22}}{E_y} - \frac{\tilde{a}_{11}}{E_x},$$

$$R(y, \xi) = \sum_{n=0}^{\infty} K_{n+1}(y, \xi),$$

$$K_1(y, \xi) = \frac{-1}{a_{11}} \int_{\xi}^y \left[\left(\frac{d^2 \beta_1(\eta)}{d\eta^2} + s^2 \beta_2(\eta) \right) \operatorname{sh} s(y - \eta) \operatorname{sh} s(\eta - \xi) + s^2 (2\beta_1(\eta) - g(\eta)) \operatorname{ch} s(y - \eta) \operatorname{sh} s(\eta - \xi) \right] d\eta,$$

$$K_{n+1}(y, \xi) = \int_{\xi}^y K_1(y, t) K_n(t, \xi) dt,$$

$$\mu(y, \xi) = [(I_{12} f_A(y) - I_{11} f_B(y)) \operatorname{ch} s\xi - (I_{22} f_A(y) - I_{21} f_B(y)) \operatorname{sh} s\xi] / \gamma.$$

By making use of solution (12), relations (4), and second equation (1), relations (10) can be given as the following system of linear algebraic equations:

$$\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} \\ \mu_{21} & \mu_{22} & \mu_{23} & \mu_{24} \\ \mu_{31} & \mu_{32} & \mu_{33} & \mu_{34} \\ \mu_{41} & \mu_{42} & \mu_{43} & \mu_{44} \end{pmatrix} \begin{pmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \bar{q}_1 \\ \bar{q}_2 \end{pmatrix} = \begin{pmatrix} is\bar{u}_1 - \tau_1 \\ is\bar{u}_2 - \tau_2 \\ is\bar{v}_1 - \tau_3 \\ is\bar{v}_2 - \tau_4 \end{pmatrix}, \quad (13)$$

where

$$\begin{aligned} \mu_{11} &= a_{11}(b) \chi_1(b) - a_{12}(b), & \mu_{12} &= a_{11}(b) \chi_2(b), \\ \mu_{13} &= a_{11}(b) \chi_3(b), & \mu_{14} &= a_{11}(b) \chi_4(b), \\ \mu_{21} &= a_{11}(-b) \chi_1(-b), & \mu_{22} &= a_{11}(-b) \chi_2(-b) - a_{12}(-b), \\ \mu_{23} &= a_{11}(-b) \chi_3(-b), & \mu_{24} &= a_{11}(-b) \chi_4(-b), \\ \mu_{31} &= \frac{i}{s} \frac{d}{dy} (a_{11}(b) \chi_1(b) + a_{12}(b)), \\ \mu_{32} &= \frac{i}{s} \frac{d}{dy} (a_{11}(b) \chi_2(b)), & \mu_{34} &= \frac{i}{s} \frac{d}{dy} (a_{11}(b) \chi_4(b)), \\ \mu_{33} &= g(b) + \frac{i}{s} \left[\frac{d}{dy} (a_{11}(b) \chi_3(b)) - is a_{12}(b) \right], \\ \mu_{41} &= \frac{i}{s} \frac{d}{dy} (a_{11}(-b) \chi_1(-b)), & \mu_{43} &= \frac{i}{s} \frac{d}{dy} (a_{11}(-b) \chi_3(-b)), \\ \mu_{42} &= \frac{i}{s} \frac{d}{dy} (a_{11}(b) \chi_2(b) - a_{12}(b)), \\ \mu_{44} &= g(-b) + \frac{i}{s} \left[\frac{d}{dy} (a_{11}(-b) \chi_4(-b)) - is a_{12}(-b) \right], \\ \tau_1 &= \alpha_1(b) \bar{T}(b) + a_{11}(b) \chi_T(b), \\ \tau_2 &= \alpha_1(-b) \bar{T}(-b) - a_{11}(-b) \chi_T(-b), \\ \tau_3 &= \frac{i}{s} \frac{d}{dy} (a_{11}(b) \chi_T(b) + \alpha_1(b) \bar{T}(b)), \\ \tau_4 &= \frac{i}{s} \frac{d}{dy} (a_{11}(-b) \chi_T(-b) + \alpha_1(-b) \bar{T}(-b)). \end{aligned}$$

System (13) presents the one-to-one relations between the stresses and displacements on the boundaries $y = \pm b$ of an orthotropic crosswise-nonhomogeneous strip. From these relations, we can find the boundary tractions in terms of the boundary displacements, and vice versa. This allows us to solve the problems with boundary conditions in terms of stresses; displacements, or ones of mixed type by reducing all of them to solution (12).

4 Numerical Example and Discussion

Consider an example of computation of the unknown boundary tractions (11) in the strip ($b=1$) with clamped sides ($u_j = v_j = 0, j=1,2$) subjected to two-dimensional distribution of temperature field with the following Fourier transformation [4]:

$$\frac{\bar{T}}{T_0} = \sqrt{\pi} \exp\left(-\frac{s^2}{4}\right) \frac{\cosh sy}{\cosh sb}.$$

The orthotropic elastic moduli have the following form [3]:

$$E_x = \frac{E_y}{\delta_1} = \text{const}, \quad \frac{\nu_{xy}}{\delta_1} = \nu_{yx} = 0.3,$$

$$G_{xy} = \frac{E_x}{2(1+\nu_{xy})} = \text{const},$$

$$\alpha_j = \alpha_j^0 (2+y)^k, \quad j=1,2, \quad \frac{\alpha_2^0}{\delta_2} = \alpha_1^0.$$

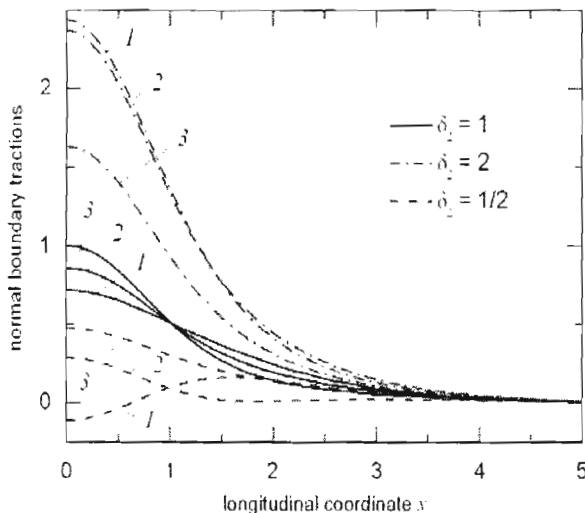


Figure 1: Dimensionless (normalized by $\alpha_1^0 E_x T_0$) normal tractions on the sides $y = \pm b$ of the clamped orthotropic nonhomogeneous strip. (Curve 1: p_1 at $k=1$; curve 2: p_2 at $k=1$; curve 3: p_1 and p_2 at $k=0$)

In Fig. 1, the normal tractions are depicted versus the longitudinal coordinate x for $\delta_1 = 1$. As we can see, for the case of homogeneous material ($k=0$), the tractions on both sides are equal for different ratio between the constant elastic moduli. We can conclude that greater values of δ_2 cause the tractions of larger magnitude at $x=0$. The anisotropy effects substantially

in the distribution of tractions for both homogeneous and nonhomogeneous cases. Note that for the latter one, when $\delta_2 = \frac{1}{2}$ (dashed curve 1), in contrast to the other cases, the traction p_1 is compressive in vicinity of $x=0$. This can be explained by the fact that longitudinal thermal expansion is double of the transversal expansion, which leads to the contractions in transversal direction.

5 Conclusion

By means of correct integration of the Cauchy relations along with an explicit solution for the problem in terms of stresses, the one-to-one relations between the stresses and displacements on the boundary of an orthotropic arbitrarily-nonhomogeneous strip are established. This allows for solution of the problems for the mentioned domain with various kinds of the boundary conditions in one manner by reducing them to the solved problem in terms of stresses.

Acknowledgments

The 1st author acknowledges the support of this research by the joint grant of the Ukrainian National Academy of Sciences and Russian Foundation of Basic Research (No 0114U005082). The 2nd and 3rd authors gratefully acknowledge particular financial support of this research by the National Science Council (ROC) under grant NSC 102-2811-E-002-018.

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