

BENDING OF NONTHIN TRANSVERSALLY ISOTROPIC PLATES WITH CURVILINEAR OPENINGS

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The problem on the spatial stress–strain state of a nonthin transversally isotropic plate with a curvilinear nearly circular opening is considered. The plate is assumed to be subject to bending moments on the opening contour and at infinity. To solve the problem, it is proposed to make mutual use of two well-know methods — expansion of the displacement and stress components into Fourier series in Legendre polynomials of the thickness coordinate and boundary-shape perturbation. The equations and recurrent relations necessary for solution of the problem in an arbitrary approximation are presented.

The conformal-mapping method [6] is used to solve problems of the classical theory of bending of plates with curvilinear openings. The boundary-shape perturbation method [2–5, 12] turns out to be efficient for the applied theories that employ the Helmholtz equations apart from the biharmonic equation. However, the applied theories that take into account lateral-shear strains are mainly used to study the stress–strain state of thin (or average-thickness) plates. The present paper is devoted to an approximate analytical approach to the study of the spatial stress–strain state of nonthin transversally isotropic plates that have curvilinear (noncircular) openings (cylindrical cavities) and are subjected to bending. This approach is based on the method of expansion of desired functions into Fourier series in Legendre polynomials of the thickness coordinate [1, 11] and the boundary-shape perturbation method [2–5]. The combination of these analytical methods was tested on the problem on the stress–strain state of a shallow spherical shell weakened by a small curvilinear opening [12].

1. Bending Equilibrium Equations for Nonthin Plates and Their General Solutions. Assume that a plate of thickness $2h$ ($h = \text{const}$) occupying the domain $\Omega = G \times [-h, h]$ in three-dimensional space is referred to a Cartesian coordinate system x_i ($i = 1, 2, 3$). Following [1, 9], we will represent the components of the displacement vector u_j and the stress tensor σ_{ij} in the form of a finite Fourier series in Legendre polynomials $P_k(\xi)$, i.e.,

$$\{u_j, \sigma_{ij}\} = \sum_{k=0}^N \left(k + \frac{1}{2} \right) \{u_j^{(k)}(x), h^{-1} \sigma_{ij}^{(k)}(x)\} P_k(\xi), \quad (1.1)$$

where $x = (x_1, x_2) \in G$, $\xi = h^{-1} x_3 \in [-1, 1]$, $u_j^{(k)}(x)$, and $\sigma_{ij}^{(k)}(x)$ are expansion coefficients called moments. For the sake of determinacy, we will consider that $N = 2n + 1$ ($n = 0, 1, \dots$).

The system of bending equilibrium equations for the plate in the moments $\sigma_{ij}^{(m)}$ has the form

$$\begin{aligned} \frac{\partial \sigma_{\alpha\beta}^{(2k+1)}}{\partial x_\alpha} - \frac{1}{h} \sum_{l=0}^k (4l+1) \sigma_{\beta 3}^{(2l)} + h f_\beta^{(2k+1)} &= 0 \quad (\beta = 1, 2), \\ \frac{\partial \sigma_{\alpha 3}^{(2k)}}{\partial x_\alpha} - \frac{1}{h} \sum_{l=0}^{k-1} (4l+3) \sigma_{33}^{(2l+1)} + h f_3^{(2k)} &= 0, \quad k = 0, 1, \dots, n, \end{aligned} \quad (1.2)$$

where $f_i^{(m)} = \frac{1}{h} [\sigma_{3i}^+ - (-1)^m \sigma_{3i}^-]$, σ_{3i}^+ and σ_{3i}^- are the stress component on the front boundary planes $x_3 = h$ and $x_3 = -h$, respectively, and summation is carried out over the index α from 1 to 2. The moments of stresses $\sigma_{ij}^{(m)}$ and displacements $u_j^{(m)}$ are related as follows:

$$\begin{aligned}\sigma_{11}^{(2k+1)} &= h \left[c_{11} \frac{\partial u_1^{(2k+1)}}{\partial x_1} + c_{12} \frac{\partial u_2^{(2k+1)}}{\partial x_2} + \frac{c_{13}}{h} \sum_{l=k+1}^n (4l+1) u_3^{(2l)} \right] \quad (1 \neq 2), \\ \sigma_{33}^{(2k+1)} &= h \left[c_{13} e^{(2k+1)} + \frac{c_{33}}{h} \sum_{l=k+1}^n (4l+1) u_3^{(2l)} \right], \\ \sigma_{12}^{(2k+1)} &= h c_{66} \left(\frac{\partial u_1^{(2k+1)}}{\partial x_2} + \frac{\partial u_2^{(2k+1)}}{\partial x_1} \right), \\ \sigma_{\alpha 3}^{(2k)} &= h c_{44} \left[\frac{\partial u_3^{(2k)}}{\partial x_\alpha} + \frac{1}{h} \sum_{l=k}^n (4l+3) u_\alpha^{(2l+1)} \right] \quad (\alpha = 1, 2, k = 0, 1, \dots, n).\end{aligned}\quad (1.3)$$

Here, c_{ij} ($i, j = 1, 2, 3$), c_{44} , and c_{66} are the elastic constants of the material, and $c_{11} = c_{22}$, $c_{13} = c_{23}$, $c_{21} = c_{12}$, $2c_{66} = c_{11} - c_{12}$, and

$$e^{(2k+1)} = \partial u_1^{(2k+1)} / \partial x_1 + \partial u_2^{(2k+1)} / \partial x_2. \quad (1.4)$$

If we substitute (1.3) into (1.2), we obtain a system of equations in the moments of displacement components. In a complex form, it can be written as

$$\begin{aligned}c_{66} \Delta u_+^{(2k+1)} + 2(c_{12} + c_{66}) \frac{\partial e^{(2k+1)}}{\partial \bar{z}} + \frac{2}{h} \sum_{l=0}^n (4l+1) \rho_{2l}^{(k)} \frac{\partial u_3^{(2l)}}{\partial \bar{z}} \\ - \frac{c_{44}}{h^2} \sum_{l=0}^n (4l+3) \alpha_{2l+1}^{(k)} u_+^{(2l+1)} + f_+^{(2k+1)} = 0 \quad (k = 0, 1, \dots, n), \\ c_{44} \Delta u_3^{(2k)} + \frac{1}{h} \sum_{l=0}^n (4l+3) \delta_{2l+1}^{(k)} e^{(2l+1)} - \frac{c_{33}}{h^2} \sum_{l=0}^n (4l+1) \beta_{2l}^{(k)} u_3^{(2l)} + f_3^{(2k)} = 0,\end{aligned}\quad (1.5)$$

where $\Delta = 4 \partial^2 / \partial z \partial \bar{z}$ is the Laplacian, $z = x_1 + i x_2$, $u_+^{(2k+1)} = u_1^{(2k+1)} + i u_2^{(2k+1)}$, and

$$\begin{aligned}\rho_{2l}^{(k)} &= \begin{cases} -c_{44}, & 0 \leq l \leq k, \\ c_{13}, & k < l \leq n, \end{cases} \quad \delta_{2l+1}^{(k)} = \begin{cases} -c_{13}, & 0 \leq l < k, \\ c_{44}, & k \leq l \leq n, \end{cases} \\ \beta_{2l}^{(k)} &= \begin{cases} l(2l+1), & 0 \leq l \leq k, \\ k(2k+1), & k \leq l \leq n, \end{cases} \quad \alpha_{2l+1}^{(k)} = \begin{cases} (l+1)(2l+1), & 0 \leq l \leq k, \\ (k+1)(2k+1), & k \leq l \leq n. \end{cases}\end{aligned}\quad (1.6)$$

The general solution of the homogeneous ($f_+^{(2k+1)} = 0$, $f_3^{(2k)} = 0$) system of equations (1.5) can be represented by the sum of the biharmonic u , potential Ω_m , and vortex ω_s solutions [10]. In the polar coordinates r and φ , it has the form

$$\begin{aligned}
c_{66} u_3^{(0)} &= u + \sum_{m=1}^{2n} c_m^{(0)} \Omega_m, \quad c_{66} u_3^{(2k)} = \delta_*^{(2k)} h^2 \Delta u + \sum_{m=1}^{2n} c_m^{(2k)} \Omega_m, \\
c_{66} u_r^{(1)} &= h \left[-\frac{1}{3} \frac{\partial u}{\partial r} - v_1^* h^2 \frac{\partial \Delta u}{\partial r} + \sum_{m=1}^{2n} a_m^{(1)} \frac{\partial \Omega_m}{\partial r} - \sum_{s=1}^{n+1} b_s^{(1)} \frac{1}{r} \frac{\partial \omega_s}{\partial \varphi} \right], \\
c_{66} u_\varphi^{(1)} &= h \left[-\frac{1}{3} \frac{1}{r} \frac{\partial u}{\partial \varphi} - v_1^* h^2 \frac{1}{r} \frac{\partial \Delta u}{\partial \varphi} + \sum_{m=1}^{2n} a_m^{(1)} \frac{1}{r} \frac{\partial \Omega_m}{\partial \varphi} + \sum_{s=1}^{n+1} b_s^{(1)} \frac{\partial \omega_s}{\partial r} \right], \\
c_{66} u_v^{(2k+1)} &= h \left[\delta_*^{(2k+1)} \frac{\partial \Delta u}{\partial r} + \sum_{m=1}^{2n} a_m^{(2k+1)} \frac{\partial \Omega_m}{\partial r} - \sum_{s=1}^{n+1} b_s^{(2k+1)} \frac{1}{r} \frac{\partial \omega_s}{\partial \varphi} \right], \\
c_{66} u_\varphi^{(2k+1)} &= h \left[\delta_*^{(2k+1)} \frac{1}{r} \frac{\partial \Delta u}{\partial \varphi} + \sum_{m=1}^{2n} a_m^{(2k+1)} \frac{1}{r} \frac{\partial \Omega_m}{\partial \varphi} + \sum_{s=1}^{n+1} b_s^{(2k+1)} \frac{\partial \omega_s}{\partial r} \right], \tag{1.7}
\end{aligned}$$

where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}$, $a_m^{(2k+1)}$, $b_m^{(2k+1)}$, and $c_m^{(2k)}$ are constants, u , Ω_m , and ω_s are the solutions of the equations

$$\begin{aligned}
\Delta \Delta u &= 0, \\
\Delta \Omega_m - k_m^2 h^{-2} \Omega_m &= 0 \quad (m = 1, 2, \dots, 2n), \\
\Delta \omega_s - \lambda_s^2 h^{-2} \omega_s &= 0 \quad (s = 1, 2, \dots, n+1), \tag{1.8}
\end{aligned}$$

in which the parameters k_m^2 and λ_s^2 , which may be real positive, negative, and complex conjugate, are the roots of the corresponding characteristic equations [11]. The constants $\delta_*^{(2k)}$ and $\delta_*^{(2k+1)}$ in (1.7) have the following values:

$$\delta_*^{(2k)} = \begin{cases} v_2^*, & k=1, \\ 0, & k>1, \end{cases} \quad \delta_*^{(2k+1)} = \begin{cases} v_3^*, & k=1, \\ 0, & k>1, \end{cases} \tag{1.9}$$

and

$$\begin{aligned}
v_1^* &= \frac{6c c_{11} c_{33} - c_{13} c_{44}}{45 c_{33} c_{44}}, \quad v_2^* = \frac{c_{13}}{15 c_{33}}, \\
v_3^* &= \frac{c c_{11} c_{33} - c_{13} c_{44}}{105 c_{33} c_{44}}, \quad c = \frac{c_{11} c_{33} - c_{13}^2}{c_{11} c_{33}}. \tag{1.10}
\end{aligned}$$

Relations (1.3) in the polar coordinates r and φ are represented by the formulas

$$\begin{aligned}
\sigma_{rr}^{(1)} &= -\frac{2h^2}{3} \frac{\partial^2 u}{\partial r^2} - 2v_1^* h^4 \frac{\partial^2 \Delta u}{\partial r^2} - a_1 h^2 \Delta u \\
&+ \sum_{m=1}^{2n} \left(d_m^{(1)} \Omega_m + 2h^2 a_m^{(1)} \frac{\partial^2 \Omega_m}{\partial r^2} \right) - 2h^2 \sum_{s=1}^{n+1} b_s^{(1)} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \omega_s}{\partial \varphi} \right),
\end{aligned}$$

$$\begin{aligned}
\sigma_{r\varphi}^{(1)} &= -\frac{2h^2}{3} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u}{\partial \varphi} \right) - 2v_1^* h^4 \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Delta u}{\partial \varphi} \right) \\
&+ 2h^2 \sum_{m=1}^{2n} a_m^{(1)} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Omega_m}{\partial \varphi} \right) + \sum_{s=1}^{n+1} b_s^{(1)} \left(2h^2 \frac{\partial^2 \omega_s}{\partial r^2} - \lambda_s^2 \omega_s \right), \\
\sigma_{\varphi\varphi}^{(1)} &= -\sigma_{rr}^{(1)} - 2a_2 h^2 \Delta u + 2 \sum_{m=1}^{2n} \tilde{d}_m^{(1)} \Omega_m, \quad \sigma_{33}^{(1)} = \sum_{m=1}^{2n} \tilde{\tilde{d}}_m^{(1)} \Omega_m, \\
\sigma_{rr}^{(2k+1)} &= \delta^{(2k+1)} h^4 \frac{\partial^2 \Delta u}{\partial r^2} + \sum_{m=1}^{2n} \left(d_m^{(2k+1)} + 2h^2 a_m^{(2k+1)} \frac{\partial^2 \Omega_m}{\partial r^2} \right) - 2h^2 \sum_{s=1}^{n+1} b_s^{(2k+1)} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \omega_s}{\partial \varphi} \right), \\
\sigma_{r\varphi}^{(2k+1)} &= \delta^{(2k+1)} h^4 \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Delta u}{\partial \varphi} \right) + 2h^2 \sum_{m=1}^{2n} a_m^{(2k+1)} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Omega_m}{\partial \varphi} \right) + \sum_{s=1}^{n+1} b_s^{(2k+1)} \left(2h^2 \frac{\partial^2 \omega_s}{\partial r^2} - \lambda_s^2 \omega_s \right), \\
\sigma_{\varphi\varphi}^{(2k+1)} &= -\sigma_{rr}^{(2k+1)} + 2 \sum_{m=1}^{2n} \tilde{d}_m^{(2k+1)} \Omega_m, \quad \sigma_{33}^{(2k+1)} = \sum_{m=1}^{2n} \tilde{\tilde{d}}_m^{(2k+1)} \Omega_m \\
&(k = 1, 2, \dots, n), \\
\sigma_{r3}^{(2k)} &= h \left[\delta^{(2k)} h^2 \frac{\partial \Delta u}{\partial r} + \sum_{m=1}^{2n} p_m^{(2k)} \frac{\partial \Omega_m}{\partial r} - \sum_{s=1}^{n+1} q_s^{(2k)} \frac{1}{r} \frac{\partial \omega_s}{\partial \varphi} \right], \\
\sigma_{\varphi 3}^{(2k)} &= h \left[\delta^{(2k)} h^2 \frac{1}{r} \frac{\partial \Delta u}{\partial \varphi} + \sum_{m=1}^{2n} p_m^{(2k)} \frac{1}{r} \frac{\partial \Omega_m}{\partial \varphi} + \sum_{s=1}^{n+1} q_s^{(2k)} \frac{\partial \omega_s}{\partial r} \right] \\
&(k = 0, 1, \dots, n),
\end{aligned} \tag{1.11}$$

where $a_1 = a - 2/3$, $a_2 = a - 1/3$, and $a = c c_{11}/3 c_{66}$,

$$\begin{aligned}
\delta^{(2k)} &= \begin{cases} -a, & k=0, \\ a/5, & k=1, \\ 0, & k>1, \end{cases} \quad \delta^{(2k+1)} = \begin{cases} 2v_3^*, & k=1, \\ 0, & k>1, \end{cases} \\
d_m^{(2k+1)} &= \frac{c_{12}}{c_{66}} a_m^{(2k+1)} k_m^2 + \frac{c_{13}}{c_{66}} \sum_{l=k+1}^n (4l+1) c_m^{(2l)}, \\
p_m^{(2k)} &= \frac{c_{44}}{c_{66}} \left[c_m^{(2k)} + \sum_{l=k}^n (4l+3) a_m^{(2l+1)} \right], \quad q_s^{(2k)} = \frac{c_{44}}{c_{66}} \sum_{l=k}^n (4l+3) b_s^{(2l+1)}.
\end{aligned} \tag{1.12}$$

The constants $\tilde{d}_m^{(2k+1)}$ and $\tilde{\tilde{d}}_m^{(2k+1)}$ follow from $d_m^{(2k+1)}$ if c_{12} is replaced by $c_{12} + c_{66}$ and c_{13} in the first term and c_{13} by c_{13} and c_{33} in the second term, respectively.

Using formulas (1.7) and (1.11), we can write boundary conditions on the contour L_0 of radius R of the circular opening. Let the components of the displacement vector $\bar{u}_j(\varphi, \xi)$ ($j = r, \varphi, 3$) be specified on the cylindrical surface

$R \times [-h, h]$. Then, expanding them into a Fourier series of the form (1.1) and equating the expansion coefficients $\bar{u}_j^{(m)}(\varphi)$ on the contour $r = R$ to the corresponding moments $u_j^{(m)}(r, \varphi)$, we obtain

$$\begin{aligned} u_{\alpha}^{(2k+1)}(r, \varphi) \Big|_{r=R} &= \bar{u}_{\alpha}^{(2k+1)}(\varphi) \quad (\alpha = r, \varphi), \\ u_3^{(2k)}(r, \varphi) \Big|_{r=R} &= \bar{u}_3^{(2k)}(\varphi), \quad k = 0, 1, \dots, n. \end{aligned} \quad (1.13)$$

Similarly, if the components of the stress vector $\bar{\sigma}_{rj}(\varphi, \xi)$ are specified on the surface $R \times [-h, h]$, we have the conditions

$$\begin{aligned} \sigma_{r\alpha}^{(2k+1)}(r, \varphi) \Big|_{r=R} &= \bar{\sigma}_{r\alpha}^{(2k+1)}(\varphi) \quad (\alpha = r, \varphi), \\ \sigma_r^{(2k)}(r, \varphi) \Big|_{r=R} &= \bar{\sigma}_r^{(2k)}(\varphi), \quad k = 0, 1, \dots, n, \end{aligned} \quad (1.14)$$

where $\bar{\sigma}_{rj}^{(m)}(\varphi)$ and $\bar{u}_j^{(m)}(\varphi)$ are the moments of the specified stress and displacement components determined by the formulas

$$\begin{aligned} \bar{\sigma}_{rj}^{(m)}(\varphi) &= \int_{-1}^1 \bar{\sigma}_{rj}(\varphi, \xi) P_m(\xi) d\xi, \\ \bar{u}_j^{(m)}(\varphi) &= \frac{1}{h} \int_{-1}^1 \bar{u}_j(\varphi, \xi) P_m(\xi) d\xi. \end{aligned} \quad (1.15)$$

Let the biharmonic function u have the form

$$u = A_0 \ln r + \sum_{n=1}^{\infty} \left[\left(\frac{A_n}{r^n} + \frac{B_n}{r^{n-2}} \right) \cos n\varphi + \left(\frac{\tilde{A}_n}{r^n} + \frac{\tilde{B}_n}{r^{n-2}} \right) \sin n\varphi \right], \quad (1.16)$$

and the functions Ω_m and ω_s (depending on the values of the roots k_m^2 and λ_s^2 of the characteristic equations) be represented as follows:

real positive roots

$$\Omega_m = \sum_{n=-\infty}^{\infty} C_n^{(m)} K_n \left(\frac{r}{h} |k_m| \right) e^{in\varphi}, \quad \omega_s = \sum_{n=-\infty}^{\infty} \tilde{C}_n^{(s)} K_n \left(\frac{r}{h} |\lambda_s| \right) e^{in\varphi}, \quad (1.17)$$

real negative roots

$$\Omega_m = \sum_{n=-\infty}^{\infty} D_n^{(m)} Y_n \left(\frac{r}{h} |k_m| \right) e^{in\varphi}, \quad \omega_s = \sum_{n=-\infty}^{\infty} \tilde{D}_n^{(s)} Y_n \left(\frac{r}{h} |\lambda_s| \right) e^{in\varphi}, \quad (1.18)$$

complex roots ($k_m^2 = k_m' - i k_m''$, $\lambda_s^2 = \lambda_s' - i \lambda_s''$)

$$\Omega_m = \sum_{n=-\infty}^{\infty} E_n^{(m)} H_n^{(1)} \left(\frac{r}{h} |k_m| e^{i\alpha_m} \right) e^{in\varphi} \quad \left(\alpha_m = \frac{1}{2} \left(\pi - \arctan \frac{k_m''}{k_m'} \right) \right)$$

$$\omega_s = \sum_{n=-\infty}^{\infty} \tilde{E}_n^{(s)} H_n^{(1)} \left(\frac{r}{h} |\lambda_s| e^{i\beta_s} \right) e^{in\varphi} \left(\beta_s = \frac{1}{2} \left(\pi - \arctan \frac{\lambda_s''}{\lambda_s'} \right) \right). \quad (1.19)$$

Here, $A_0, A_n, \dots, \tilde{E}_n^{(s)}$ are arbitrary constants to be determined from the boundary conditions (1.13) and (1.14) and K_n, Y_n , and $H_n^{(1)}$ are the cylindrical MacDonald, Neumann, and Hankel functions. It should be noted that functions damped at infinity are used in representations (1.17)–(1.19).

2. An Analytical Method for Solution of the Problem on Bending of a Plate with a Curvilinear Opening. Based on the solution found, we will consider the problem on the stress–strain state of a nonthin transversally isotropic plate with a curvilinear opening whose contour L differs slightly from a circle L_0 contour of radius R . To solve it, let us make use of the efficient approximate boundary-shape perturbation method proposed in [2] as applied to the problem of determination of the stress concentration near curvilinear openings in thin shells. Subsequently, this method was extended to three-dimensional problems of the theory of elasticity for orthogonal noncanonical domains [3, 7], for homogeneous bodies with noncanonical interfaces [8], etc. The development and generalization of the method [2] to problems of continuum mechanics are analyzed in the survey [4] and the monographs [5, 8].

Let us assume that the smooth contour L of the curvilinear opening is described by the equations

$$x = \rho \cos \vartheta + \varepsilon \rho^{-l} \cos l \vartheta, \quad y = \rho \sin \vartheta - \varepsilon \rho^{-l} \sin l \vartheta, \quad (2.1)$$

where $x = x_1/R$ and $y = x_2/R$ are dimensionless Cartesian coordinates, ρ and ϑ are orthogonal curvilinear coordinates, and ε and l are parameters characterizing the shape of the opening. In complex form, Eqs. (2.1) are represented as follows:

$$x + iy = \omega(\zeta) = \zeta + \varepsilon f(\zeta) = r e^{i\varphi}, \quad (2.2)$$

where $\zeta = \rho e^{i\vartheta}$, $f(\zeta) = \rho^{-l} e^{-il\vartheta}$.

Since the coordinate system (ρ, ϑ, ξ) is turned at the angle β about the common axis $\xi = x_3/h$ relative to (r, φ, x_3) , the stresses $\sigma_{m'j'}$ and the displacements $u_{j'}$ can be written with the help of formulas used for conversion of the components of the stress tensor $\{\sigma\}$, the displacement vector $\{u\}$, and the scalar function ϕ in passing from the variables r, φ , and x_3 to ρ, ϑ , and ξ , i.e.,

$$\sigma_{i'j'} = \sum_{k=1}^3 \sum_{s=1}^3 \lambda_{i'k} \lambda_{j's} \sigma_{ks}, \quad u_{j'} = \sum_{k=1}^3 \lambda_{j'k} u_k, \quad \phi' = \phi$$

$$(1, 2, 3 \sim r, \varphi, x_3; 1', 2', 3' \sim \rho, \vartheta, \xi). \quad (2.3)$$

In Eqs. (2.3), the following dependences hold:

$$\sigma_{i'j'} = \sigma_{i'j'}(\rho, \vartheta, \xi), \quad u_{j'} = u_{j'}(\rho, \vartheta, \xi), \quad \phi' = \phi'(\rho, \vartheta, \xi),$$

$$\sigma_{ks} = \sigma_{ks}(r, \varphi, x_3), \quad u_k = u_k(r, \varphi, x_3), \quad \phi = \phi(r, \varphi, x_3). \quad (2.4)$$

For the transformation of coordinates being considered, the direction cosines $\lambda_{j's}$ take the values

$$\lambda_{1'1} = \lambda_{2'2} = \cos \beta, \quad \lambda_{1'2} = -\lambda_{2'1} = \sin \beta, \quad \lambda_{3'3} = 1,$$

$$\lambda_{1'3} = \lambda_{2'3} = 0, \quad \lambda_{3'1} = \lambda_{3'2} = 0. \quad (2.5)$$

According to Eqs. (2.2), the variables r, φ, x_3 and ρ, ϑ, ξ are related as follows:

$$r = |\omega(\zeta)|, \quad \varphi = \arctan \frac{I_m \omega(\zeta)}{\operatorname{Re} \omega(\zeta)}, \quad e^{i\beta} = \frac{\zeta}{|\zeta|} \frac{\omega'(\zeta)}{|\omega'(\zeta)|} \cdot \frac{\overline{\omega(\zeta)}}{|\omega(\zeta)|}. \quad (2.6)$$

Since the coordinate $\xi = x_3 h^{-1}$ is common for the systems introduced above, we have expansions in the variables ρ , ϑ , and ξ similar to (1.1), i.e.,

$$\{u_{j'}, \sigma_{i'j'}\} = \sum_{m=0}^N \left(m + \frac{1}{2}\right) \{u_j^{(m)}(\rho, \vartheta), h^{-1} \sigma_{i'j'}^{(m)}(\rho, \vartheta)\} P_m(\xi). \quad (2.7)$$

From here, we obtain the following boundary conditions on the contour L ($\rho = 1$):

(a) for the displacements $\bar{u}_{j'}$ given on the surface $L \times [-h, h]$,

$$u_{i'}^{(2k+1)}(\rho, \vartheta) \Big|_{\rho=1} = \bar{u}_{i'}^{(2k+1)}(\vartheta) \quad (i' = \rho, \vartheta), \quad u_{\xi}^{(2k)} = \bar{u}_{\xi}^{(2k)}(\vartheta), \quad (2.8)$$

(b) for the stresses $\bar{\sigma}_{\rho j'}$ given on $L \times [-h, h]$,

$$\sigma_{i'}^{(2k+1)}(\rho, \vartheta) \Big|_{\rho=1} = \bar{\sigma}_{i'}^{(2k+1)}(\vartheta) \quad (i' = \rho, \vartheta),$$

$$\sigma_{\rho}^{(2k)}(\rho, \vartheta) \Big|_{\rho=1} = \bar{\sigma}_{\rho}^{(2k)}(\vartheta). \quad (2.9)$$

Here, $\bar{u}_j^{(m)}(\vartheta)$ and $\bar{\sigma}_{\rho j'}^{(m)}(\vartheta)$ are functions given on L and determined by formulas (1.15).

It should be noted that Eqs. (1.8) in the variables ρ and ϑ in view of (2.6) will be rather complex and it is not possible to find their solution with separated variables. Therefore, following [2–5], we will search for the solutions of the above boundary-value problems as series in positive powers of a small parameter ε , i.e.,

$$\sigma_{i'j'}^{(m)} = \sum_{\tau=0}^{\infty} \varepsilon^{\tau} \sigma_{i'j'}^{(m, \tau)}, \quad u_{j'}^{(m)} = \sum_{\tau=0}^{\infty} \varepsilon^{\tau} u_{j'}^{(m, \tau)} \quad (i', j' = \rho, \vartheta, \xi). \quad (2.10)$$

On the basis of (2.6), the exponents $e^{i\beta}$ and $e^{2i\beta}$ are determined by the formulas

$$e^{i\beta} = \sum_{m=0}^{\infty} \varepsilon^m Q_m(\zeta), \quad e^{2i\beta} = \sum_{m=0}^{\infty} \varepsilon^m q_m(\zeta), \quad (2.11)$$

where, according to the rule of multiplication of series in the sense of Cauchy, we have

$$a_m(\zeta) = \sum_{n=0}^{\infty} Q_n(\zeta) Q_{m-n}(\zeta), \quad (2.12)$$

and an arbitrary scalar function is represented by the series

$$\phi^{(k)}(r, \varphi) = \sum_{n=0}^{\infty} \varepsilon^n \frac{1}{n!} L^{(n)} \phi^{(n)}(\rho, \vartheta). \quad (2.13)$$

Note that the terms of series (2.13) are found by acting by the differential operators $L^{(n)}$ on the corresponding functions $\phi^{(k)}(\rho, \vartheta)$, which are derived from $\phi^{(k)}(r, \varphi)$ by formal replacement of the variables r and φ by ρ and ϑ . The explicit form of the operators $L^{(n)}$ and the functional coefficients $Q_m(\zeta)$ are presented in [5].

Similarly to (2.13), we can represent the moments of the component of the stresses $\sigma_{ij}^{(m)}(r, \varphi)$ and displacements $u_j^{(m)}(r, \varphi)$ by formulas (2.6) as complex functions of the curvilinear coordinates ρ and ϑ . Introducing the above-mentioned expansions into the conversion formulas (2.3), allowing for the values of the direction cosines obtained from (2.5) and (2.11), and equating the expressions with the identical powers of the parameter ε , we obtain the recurrent relations

$$\begin{aligned}\sigma_{\rho\rho}^{(m, \tau)} &= \sum_{l=0}^{\tau} [\Lambda_1^{(\tau-l)} \sigma_{rr}^{(m, l)} + \Lambda_2^{(\tau-l)} (\sigma_{\varphi\varphi}^{(m, l)} - \sigma_{rr}^{(m, l)}) + \Lambda_3^{(\tau-l)} \sigma_{r\varphi}^{(m, l)}], \\ \sigma_{\vartheta\vartheta}^{(m, \tau)} &= -\sigma_{\rho\rho}^{(m, \tau)} + \sum_{l=0}^{\tau} \Lambda_1^{(\tau-l)} (\sigma_{rr}^{(m, l)} + \sigma_{\varphi\varphi}^{(m, l)}), \quad \sigma_{\xi\xi}^{(m, \tau)} = \sum_{l=0}^{\tau} \Lambda_1^{(\tau-l)} \sigma_{33}^{(m, l)}, \\ \sigma_{\rho\vartheta}^{(m, \tau)} &= \sum_{l=0}^{\tau} [\Lambda_4^{(\tau-l)} \sigma_{r\varphi}^{(m, l)} + \frac{1}{2} \Lambda_3^{(\tau-l)} (\sigma_{\varphi\varphi}^{(m, l)} - \sigma_{rr}^{(m, l)})], \quad m = 2k + 1, \\ \sigma_{\rho\xi}^{(2k, \tau)} &= \sum_{l=0}^{\tau} \left(\Lambda_5^{(\tau-l)} \sigma_{r3}^{(2k, l)} \pm \Lambda_6^{(\tau-l)} \sigma_{\varphi3}^{(2k, l)} \right) \\ \sigma_{\vartheta\xi}^{(2k, \tau)} &= \sum_{l=0}^{\tau} \left(\Lambda_5^{(\tau-l)} \sigma_{r3}^{(2k, l)} \pm \Lambda_6^{(\tau-l)} \sigma_{\varphi3}^{(2k, l)} \right)\end{aligned}\quad (2.14)$$

for the moments of displacement components

$$\begin{aligned}\frac{u_{\rho}^{(m, \tau)}}{u_{\vartheta}^{(m, \tau)}} &= \sum_{l=0}^{\tau} \left(\Lambda_5^{(\tau-l)} \frac{u_r^{(m, l)}}{u_{\varphi}^{(m, l)}} \pm \Lambda_6^{(\tau-l)} \frac{u_{\varphi}^{(m, l)}}{u_r^{(m, l)}} \right) \quad m = 2k + 1, \\ u_{\xi}^{(2k, \tau)} &= \sum_{l=0}^{\tau} \Lambda_1^{(\tau-l)} u_3^{(2k, l)},\end{aligned}\quad (2.15)$$

where $\Lambda_k^{(n)}$ ($k = 1, 2, \dots, 6$) are differential operators of the form

$$\begin{aligned}\Lambda_1^{(\tau)} &= \frac{1}{r!} L^{(\tau)}, \quad \Lambda_2^{(\tau)} = -\frac{1}{2} \sum_{s=1}^{\tau} \operatorname{Re} q_s \frac{L^{(\tau-s)}}{(\tau-s)!}, \\ \Lambda_3^{(\tau)} &= \sum_{s=1}^{\tau} \operatorname{Im} q_s \frac{L^{(\tau-s)}}{(\tau-s)!}, \quad \Lambda_4^{(\tau)} = \Lambda_1^{(\tau)} - 2 \Lambda_2^{(\tau)}, \\ \Lambda_5^{(\tau)} &= \sum_{s=0}^{\tau} \operatorname{Re} Q_s \frac{L^{(\tau-s)}}{(\tau-s)!}, \quad \Lambda_6^{(\tau)} = \sum_{s=0}^{\tau} \operatorname{Im} Q_s \frac{L^{(\tau-s)}}{(\tau-s)!}.\end{aligned}\quad (2.16)$$

From Eqs. (2.13), it follows that the moments of stresses and displacements appearing on the right-hand sides of relations (2.14) and (2.15) are written based on their analytical expressions (1.7) and (1.11) in the variables r and φ by formally replacing these variables by ρ and ϑ . For example,

$$\begin{aligned} \sigma_{\rho\rho}^{(1,l)} = & -\frac{2h^2}{3R^2} \frac{\partial^2 u^{(l)}}{\partial \rho^2} - \frac{2\nu_1^* h^4}{R^4} \frac{\partial^2 \Delta u^{(l)}}{\partial \rho^2} - a_1 \frac{h^2}{R^2} \Delta u^{(l)} \\ & + \sum_{m=1}^{2n} \left(d_m^{(1)} \Omega_m^{(1)} + 2a_m^{(1)} \frac{h^2}{R^2} \frac{\partial^2 \Omega_m^{(l)}}{\partial \rho^2} \right) - \frac{2h^2}{R^2} \sum_{s=1}^{n+1} b_s^{(1)} \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial \omega_s^{(l)}}{\partial \vartheta} \right), \end{aligned} \quad (2.17)$$

where the functions $u^{(l)}$, $\Omega_m^{(l)}$, and $\omega_s^{(l)}$ are determined by the equations

$$\Delta \Delta u^{(l)} = 0, \quad \Delta \Omega_m^{(l)} - \kappa_m^2 \Omega_m^{(l)} = 0, \quad \Delta \omega_s^{(l)} - \mu_s^2 \omega_s^{(l)} = 0, \quad (2.18)$$

where

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \vartheta^2}, \quad \kappa_m^2 = k_{bm}^2 R^2 h^{-2}, \quad \mu_s^2 = \lambda_s^2 R^2 h^{-2}. \quad (2.19)$$

Hence, the solutions of Eqs. (2.18) corresponding to representations (1.16)–(1.17) for the external problem may be accepted in the form

$$\begin{aligned} u^{(l)} = & A_0^{(l)} \ln \rho + \sum_{n=1}^{\infty} \left[\left(\frac{A_n^{(l)}}{\rho^n} + \frac{B_n^{(l)}}{\rho^{n-2}} \right) \cos n \vartheta + \left(\frac{\tilde{A}_n^{(l)}}{\rho^n} + \frac{\tilde{B}_n^{(l)}}{\rho^{n-2}} \right) \sin n \vartheta \right], \\ \Omega_m^{(l)} = & \sum_{n=-\infty}^{\infty} C_n^{(m,l)} K_n(|\kappa_m| \rho) e^{in\vartheta}, \quad \Omega_s^{(l)} = \sum_{n=-\infty}^{\infty} \tilde{C}_n^{(s,l)} K_n(|\mu_s| \rho) e^{in\vartheta}. \end{aligned} \quad (2.20)$$

The constants $A_n^{(l)}$, $\tilde{A}_n^{(l)}$, $B_n^{(l)}$, $\tilde{B}_n^{(l)}$, $C_n^{(m,l)}$, and $\tilde{C}_n^{(s,l)}$ are determined from the corresponding boundary conditions in the l -approximation. For the first basic problem, where the moments of the stress components are given on the contour of the opening, the boundary conditions have the form

$$\begin{aligned} \sigma_{\rho\rho}^{*(2k+1)} \Big|_L & \equiv (\sigma_{\rho\rho}^{0(2k+1)} + \sigma_{\rho\rho}^{(2k+1)}) \Big|_L = \bar{\sigma}_{\rho\rho}^{(2k+1)}(\vartheta, \varepsilon), \\ \sigma_{\rho\vartheta}^{*(2k+1)} \Big|_L & \equiv (\sigma_{\rho\vartheta}^{0(2k+1)} + \sigma_{\rho\vartheta}^{(2k+1)}) \Big|_L = \bar{\sigma}_{\rho\vartheta}^{(2k+1)}(\vartheta, \varepsilon), \\ \sigma_{\rho\xi}^{*(2k)} \Big|_L & \equiv (\sigma_{\rho\xi}^{0(2k)} + \sigma_{\rho\xi}^{(2k)}) \Big|_L = \bar{\sigma}_{\rho\xi}^{(2k)}(\vartheta, \varepsilon). \end{aligned} \quad (2.21)$$

Here, $\sigma_{\rho\rho}^{0(2k+1)}$, $\sigma_{\rho\vartheta}^{0(2k+1)}$, and $\sigma_{\rho\xi}^{0(2k)}$ are the components of the principal stress state and $\sigma_{\rho\rho}^{(2k+1)}$, $\sigma_{\rho\vartheta}^{(2k+1)}$, and $\sigma_{\rho\xi}^{(2k)}$ are the components of the perturbed stress state due to the presence of the opening.

It is obvious that in solving the boundary problem formulated, it is necessary, by analogy with formulas (2.10), to represent the components of the principal stress state and the given functions $\bar{\sigma}_{\rho\rho}^{(2k+1)}(\vartheta, \varepsilon)$, $\bar{\sigma}_{\rho\vartheta}^{(2k+1)}(\vartheta, \varepsilon)$, and $\bar{\sigma}_{\rho\xi}^{(2k+1)}(\vartheta, \varepsilon)$ as series in the small parameter ε .

In particular, if bending moments $\sigma_{11}^{(1)\infty} = M_1$ and $\sigma_{22}^{(1)\infty} = M_2$ ($\sigma_{12}^{(1)\infty} = 0$) are applied to the plate at infinity, then the nonzero components of the principal stress state in the coordinates ρ and ϑ represented in series in the parameter ε have the form [12]

$$\sigma_{\rho\rho}^{0(1)} = \frac{1}{2} (M_1 + M_2) + \frac{1}{2} (M_1 - M_2) \left\{ \cos 2 \vartheta \right.$$

$$\begin{aligned}
& + \frac{\varepsilon l}{\rho^{l+1}} [\cos (l+3) \vartheta - \cos (l-1) \vartheta] + \frac{\varepsilon^2 l^2}{\rho^{2(l+1)}} [\cos 2(l+2) \vartheta - \cos 2 \vartheta] + \dots \Big\}, \\
& \sigma_{\vartheta\vartheta}^{0(1)} = -\frac{1}{2} (M_1 - M_2) \Big\{ \sin 2 \vartheta + \frac{\varepsilon l}{\rho^{l+1}} [\sin (l+3) \vartheta + \sin (l-1) \vartheta] \\
& \quad + \frac{\varepsilon^2 l^2}{\rho^{2(l+1)}} [\sin 2(l+2) \vartheta - \sin 2 \vartheta] + \dots \Big\}, \\
& \sigma_{\vartheta\vartheta}^{0(1)} = M_1 + M_2 - \sigma_{\rho\rho}^{0(1)}, \quad \sigma_{\rho\xi}^{0(2k)} = \sigma_{\vartheta\xi}^{0(2k)} = 0, \quad k = 0, 1, \dots, n, \\
& \sigma_{\rho\rho}^{0(2k+1)} = \sigma_{\vartheta\vartheta}^{0(2k+1)} = \sigma_{\rho\vartheta}^{0(2k+1)} = 0, \quad k = 1, 2, \dots, n.
\end{aligned} \tag{2.22}$$

Thus, if values of the moments of the stress components are known on the contour of the opening and at infinity, then from boundary conditions (2.21), we obtain a system of algebraic equations for determination of the unknown coefficients appearing in function (2.20).

In summary, note that the analytical method stated allows us to determine the three-dimensional stress-strain state of nonthin transversally isotropic plates with curvilinear openings, mainly with prescribed accuracy. Therefore, the problem may be solved through either increase in the number of terms of the Fourier series in Legendre polynomials of the thickness coordinate or increase in the number of terms of the series in powers of a small parameter.

REFERENCES

1. I. N. Vekua, *Some General Methods for Construction of Various Modifications of the Theory of Shells* [in Russian], Nauka, Moscow (1982).
2. A. N. Guz', "An approximate method for determination of the stress concentration near curvilinear openings in shells," *Prikl. Mekh.*, **8**, No. 6, 605–612 (1962).
3. A. N. Guz', "One method for solving three-dimensional linear problems of continuum mechanics for noncanonical domains," *Dop. Akad. Nauk Ukr. SSR, Ser. A*, No. 4, 352–355 (1970).
4. A. N. Guz' and Yu. N. Nemish, "The boundary-shape perturbation method in continuum mechanics (review)," *Prikl. Mekh.*, **23**, No. 9, 3–29 (1987).
5. A. N. Guz' and Yu. N. Nemish, *The Boundary-Shape Perturbation Method in Continuum Mechanics* [in Russian], Vyshcha Shkola, Kiev (1989).
6. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity* [in Russian], Nauka, Moscow (1966).
7. Yu. N. Nemish, "An approximate solution of spatial problems of the theory of elasticity for a transversally isotropic medium," *Prikl. Mekh.*, **5**, No. 8, 26–34 (1969).
8. Yu. N. Nemish, *Elements of the Mechanics of Piecewise Homogeneous Bodies with Noncanonical Interfaces* [in Russian], Naukova Dumka, Kiev (1989).
9. Yu. N. Nemish and I. Yu. Khoma, *Nonthin Transversally Isotropic Shells*, Chapter 10, Vol.7 of the 12-volume series *The Mechanics of Composites* [in Russian], A.S.K., Kiev (1998).
10. I. Yu. Khoma, "The issue of bending of thick anisotropic plates with low shear rigidity," in: *Proc. 9th All-Union Conf. on the Theory of Shells and Plates* (Leningrad, 1973) [in Russian], Sudostroenie, Leningrad (1975), pp. 96–99.
11. I. Yu. Khoma, *The Generalized Theory of Anisotropic Shells* [in Russian], Naukova Dumka, Kiev (1986).
12. Yu. N. Nemish and I. Yu. Khoma, "Determination of the stressed-strained state of thick transversally isotropic spherical shells with curvilinear openings," *Int. Appl. Mech.*, **34**, No. 6, 552–560 (1998).