

STRESSED STATE OF ANISOTROPIC PLATE WEAKENED BY A STRENGTHENED HOLE AND A SYSTEM OF CRACKS

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Cut edges are often strengthened by elastic elements, which, being by weight an insignificant part of the construction, improve considerably static and fatigue strength. Many investigations are devoted to the analysis of stressed-strained state near a strengthened cut in unimpaired plates or to a crack extension near an unstrengthened hole. In spite of the great practical importance, the effect of strengthening ring and shapes of the cut on the value of stress intensity factors (SIF) in crack apexes has not been discussed.

Below we present particular solutions of the problem of determining stressed-strained state (SSS) of anisotropic plate weakened by a system of through cuts (cracks) and elliptic hole with a strengthened edge. The problem is reduced to a system of singular integral equations (SIE) and the algorithm of its numerical solution is given. We suggest the results of calculations illustrating the effect of the plate material anisotropy, rigidity of the strengthening ring, and shape of the cut on the stress distribution and SIF in the crack apexes.

Let an infinite rectilinearly anisotropic plate of thickness h be weakened by an elliptic hole $L_0 = \{(x/a)^2 + (y/b)^2 = 1\}$ and a system of smooth curvilinear cuts L_j ($j = \overline{1, k}$). The whole contour L_0 is strengthened by a continuously attached thin elastic ring of variable rigidity. The strengthener is acted upon by the arbitrarily distributed normal and shear loads of intensity $p_1^*(t)$, $p_2^*(t)$ ($t \in L_0$) (for $p_2^*(t)$ the action is counterclockwise, Fig. 1). The plate is subjected to a given system of external forces at infinity and the cut edges are unloaded. We think that the plate is in a generalized flat stressed state and one of the principal axes of inertia of the ring cross section lies in the median plane xoy of the plate.

We denote by $p_1(t)$, $p_2(t)$ the normal and shear contact forces arising along the plate contour L_0 from the strengthening ring (positive direction for $p_1(t)$ ($p_2(t)$) along $\vec{n}(\vec{\tau})$, Fig. 1). Using the solution on the action of a concentrated force in anisotropic infinite plate with an elliptic hole and following [6, 4], we may take general presentations of the solution for the plate in the following form

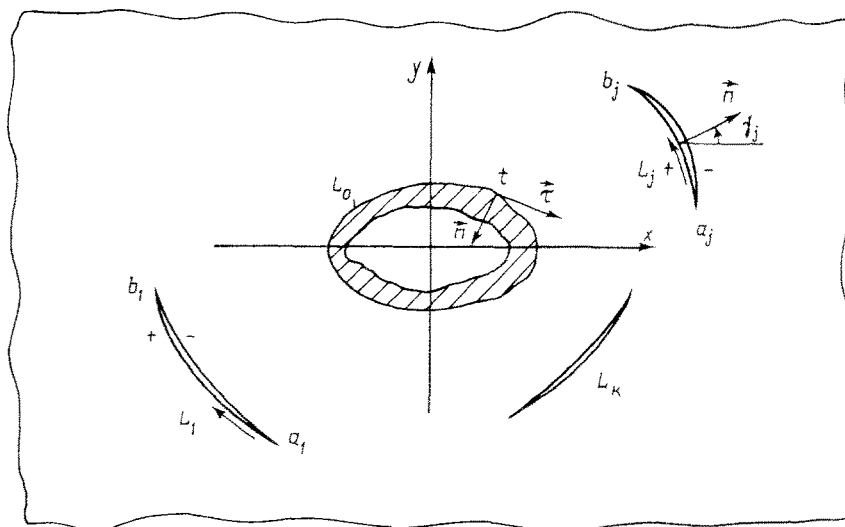


Fig. 1

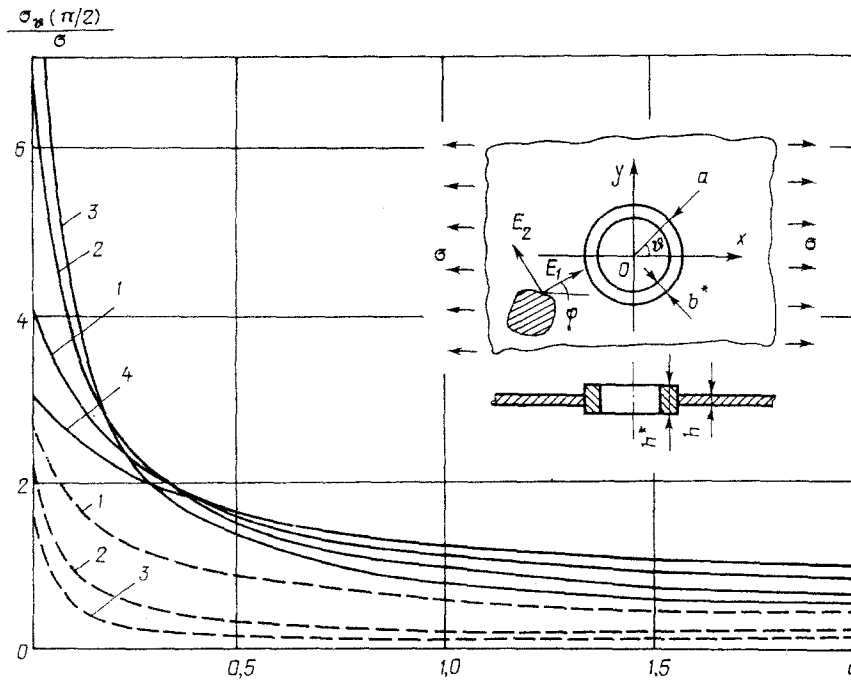


Fig. 2

$$\Phi_{\nu}(z_{\nu}) = \sum_{j=0}^3 \Phi_{\nu j}(z_{\nu}),$$

$$\Phi_{\nu k}(z_{\nu}) = \frac{1}{\omega_{\nu}'(\zeta_{\nu})} \int_{L_0} \left\{ A_{\nu}^k(\tau) + [l_{\nu} \overline{A_1^k(\tau)} + n_{\nu} \overline{A_2^k(\tau)}] \times \right.$$

$$\left. \times \frac{\eta}{\zeta_{\nu}} \right\} \frac{p_k(\tau) ds}{\zeta_{\nu} - \eta} = \frac{1}{\omega_{\nu}'(\zeta_{\nu})} \int_{L_0} P_{\nu k}(z_{\nu}, \tau) \frac{p_k(\tau) ds}{\zeta_{\nu} - \eta},$$

$$\Phi_{\nu 3}(z_{\nu}) = \frac{1}{2\pi i \omega_{\nu}'(\zeta_{\nu})} \int_{L_0} \frac{\omega_{\nu}(\tau) d\tau_{\nu}}{\eta_{\nu} - \zeta_{\nu}} + R_{\nu}(\zeta_{\nu}, \tau) \times$$

$$\times \overline{\omega_1(\tau)} d\overline{\tau}_1 + Q_{\nu}(\zeta_{\nu}, \tau) \overline{\omega_2(\tau)} d\overline{\tau}_2 \quad (k=1, 2),$$

$$R_{\nu}(\zeta_{\nu}, \tau) = \frac{l_{\nu}}{\zeta_{\nu}(1 - \zeta_{\nu} \overline{\eta}_1)}; \quad Q_{\nu}(\zeta_{\nu}, \tau) = \frac{n_{\nu}}{\zeta_{\nu}(1 - \zeta_{\nu} \overline{\eta}_2)};$$

$$z_{\nu} = \omega_{\nu}(\zeta_{\nu}) = \frac{a - i\mu_{\nu} b}{2} \zeta_{\nu} + \frac{a + i\mu_{\nu} b}{2} \zeta_{\nu}^{-1};$$

$$a(t) \omega_1(t) + b(t) \overline{\omega_1(t)} + \omega_2(t) = 0; \quad l_{\nu} = \frac{\overline{\mu}_1 - \mu_{3-\nu}}{\mu_{\nu} - \mu_{3-\nu}},$$

$$n_{\nu} = \frac{\overline{\mu}_2 - \mu_{3-\nu}}{\mu_{\nu} - \mu_{3-\nu}}; \quad L = \bigcup_{j=1}^{\infty} L_j.$$

Here and further we use the notation of [4]; $\Phi_{\nu 0}(z_{\nu})$ is the solution for a solid infinite plate with a free elliptic hole loaded by the known forces at infinity.

Thus, the selected $\Phi_{\nu}(z_{\nu})$ satisfy automatically the boundary conditions on L_0

$$h\sigma_n^-(t) = p_1(t); \quad h\tau_n^-(t) = p_2(t); \quad \varepsilon^-(t) = \varepsilon_0(t), \quad (2)$$

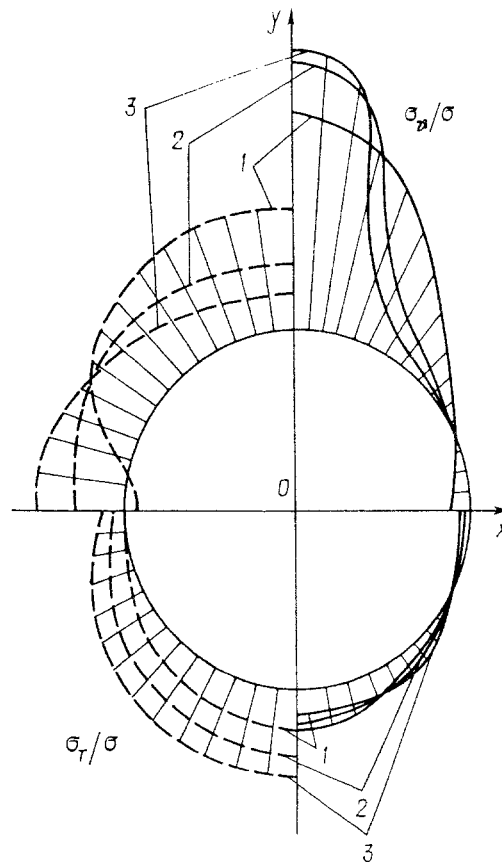


Fig. 3

where $\sigma_n^-(t)$, $\tau_n^-(t)$, $\varepsilon^-(t)$ are the limiting values on the right of the normal and shear stresses and relative elongation of the plate's arc element at the point $t \in L_0$; $\varepsilon_0(t)$ is the relative elongation of the arc element on the ring external contour (L_0) at the point $t \in L_0$.

Let us denote by $\Theta_0(t)$ the vector $\bar{\tau}(t)$ angle of rotation upon deformation. Assuming that the ring operates as a curvilinear bar and its width changes smoothly along L_0 , we may write the equation of compatibility of ring and plate deformations along L_0 in the following form [7]

$$\begin{aligned} & \sum_{v=1}^2 \left\{ (p_v + i q_v) M_v(t) \Phi_v(t_v) + (\bar{p}_v + i \bar{q}_v) \overline{M_v(t)} \overline{\Phi_v(t_v)} \right\} = \\ & \exp[i\psi(t)] \left\{ \Theta_0(t) + i \varepsilon_0(t) \right\}, \\ & \Theta_0(t) = \frac{1}{E_0} \sum_{k=1}^2 \left[\int_0^{\psi(t)} \left\{ T_k(t, \tau) p_k(\tau) + T_k^*(t, \tau) p_k^*(\tau) \right\} \times \right. \\ & \quad \left. \times d\psi(\tau) \right] + \sum_{i=1}^4 g_i V_i(t); \\ & \varepsilon_0(t) = \frac{1}{E_0} \sum_{k=1}^2 \left[\int_0^{\psi(t)} \left\{ E_k(t, \tau) p_k(\tau) + E_k^*(t, \tau) p_k^*(\tau) \right\} \times \right. \\ & \quad \left. \times d\psi(\tau) \right] + \sum_{i=1}^4 g_i U_i(t); \\ & M_v(t) = \mu_v \cos \psi(t) - \sin \psi(t), \end{aligned} \quad (3)$$

where $\psi(t)$ is the angle between the internal normal n^{-1} to the contour L_0 at a point t and the x axis, which is counted from the latter counterclockwise; E_0 is the Young modulus of the ring material; g_i ($i = \overline{1, 4}$) are the unknown integration constants.

TABLE 1

η	0,1		0,5		0,9	
N	$k_1(-1)$	$k_1(1)$	$k_1(-1)$	$k_1(1)$	$k_1(-1)$	$k_1(1)$
10	0,8488	0,8637	0,7726	0,8721	0,2940	0,8413
20	0,8508	0,8651	0,7846	0,8746	0,4086	0,8516
30	0,8510	0,8653	0,7851	0,8747	0,4609	0,8550

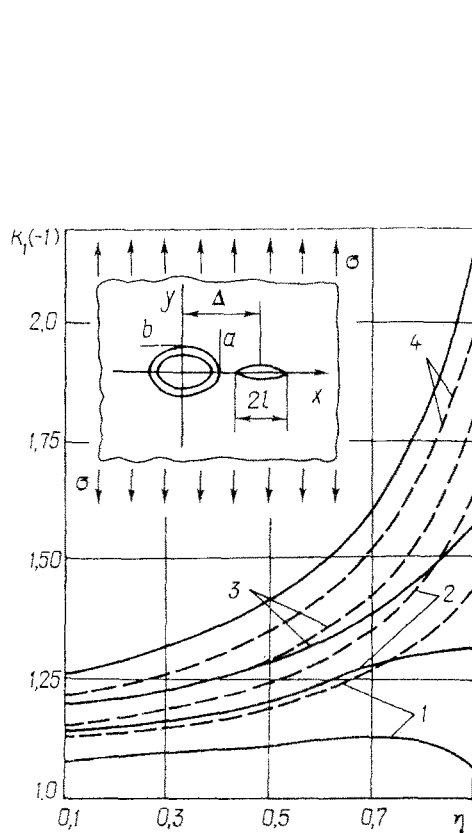


Fig. 4

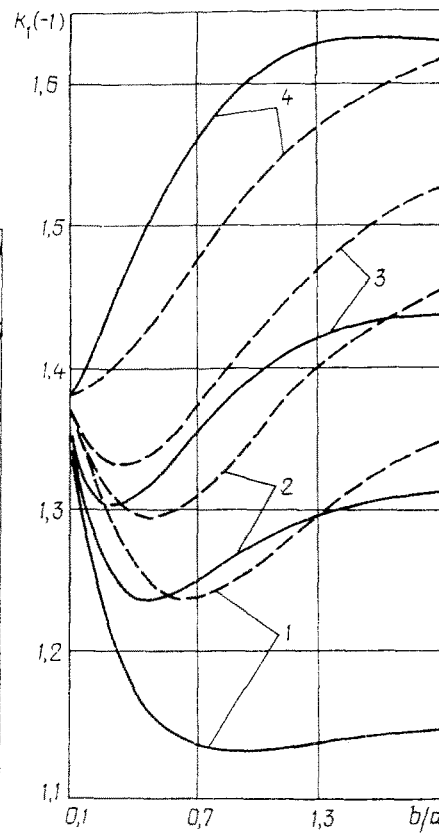


Fig. 5

An explicit form of the expressions $E_k(t, \tau)$, $E_k^*(t, \tau)$, $T_k(t, \tau)$, $T_k^*(t, \tau)$ ($k = 1, 2$), $U_i(t)$, $V_i(t)$ ($i = \overline{1, 4}$) is not given here due to its awkwardness.

Substituting (1) into the boundary conditions (3), after certain transformations we find ($\omega(\tau) = \omega_1(\tau)$)

$$\sum_{j=1}^2 \int_{L_0} \left[\frac{f_j(t, \tau)}{\tau - t} + K_{1j}(t, \tau) \right] p_j(\tau) ds + \int_L \left\{ K_{13}(t, \tau) \omega(\tau) + K_{14}(t, \tau) \overline{\omega(\tau)} \right\} ds + \sum_{s=1}^4 g_s W_s^*(t) = f_1^*(t), \quad t \in L_0 \quad (4)$$

Substituting (1) into the boundary conditions on L [4]

$$a(t) \Phi_1^+(t) + b(t) \overline{\Phi_1^+(t)} + \Phi_2^+(t) = 0,$$

we obtain the following singular integral equation (SIE):

$$\int_L \frac{\omega(\tau_1) d\tau_1}{\tau_1 - t_1} + \sum_{j=1}^2 \int_{L_0} K_{2j}(t, \tau) p_j(\tau) ds + \int_L \left\{ K_{23}(t, \tau) \omega(\tau) + \right.$$

$$\begin{aligned}
& + K_{24}(t, \tau) \overline{\omega(\tau)} \} ds = f_2^*(t), \quad t \in L. \\
K_{2j}(t, \tau) &= \frac{\pi i \omega_1'(\xi_1)}{b(t)} \left\{ \frac{\overline{a(t)}}{\omega_1(\xi_1)} \cdot \frac{\overline{P_{1j}(t, \tau)}}{\xi_1 - \bar{\eta}} + \right. \\
& \left. + \frac{\overline{b(t)}}{\omega_1(\xi_1)} \cdot \frac{\overline{P_{1j}(t, \tau)}}{\xi_1 - \eta} + \frac{1}{\omega_2(\xi_2)} \cdot \frac{\overline{P_{2j}(t, \tau)}}{\xi_2 - \bar{\eta}} \right\}, \\
f_j(t, \tau) &= \frac{\tau - t}{\xi - \eta} \sum_{v=1}^2 \{ L_v(t) P_{vj}(t, \tau) - S_v(t) \overline{P_{vj}(t, \tau)} \xi \eta \}, \\
f_1^* &= \frac{\exp(i\psi(t))}{E_0} \sum_{j=1}^2 \int_{L_0} [-T_j^*(t, \tau) + iE_j^*(t, \tau)] \times \\
& \times p_j^*(\tau) ds - \sum_{v=1}^2 L_v(t) \omega_v'(\xi_v) \Phi_{v0}(t_v) + S_v(t) \times \\
& \times \omega_v'(\xi_v) \overline{\Phi_{v0}(t_v)}, \\
f_2^*(t) &= -[a(t) \Phi_{10}(t_1) + b(t) \overline{\Phi_{10}(t_1)} + \Phi_{20}(t_2)], \\
W_j^*(t) &= \exp(i\psi(t)) [V_s(t) - iU_s(t)], \quad \xi = \xi_v(t), \quad \eta = \xi_v(\tau).
\end{aligned} \tag{5}$$

Because of awkwardness an explicit form of continuous functions $K_{1j}(t, \tau)$, $K_{j3}(t, \tau)$, $K_{j4}(t, \tau)$, $L_j(t)$, $S_j(t)$, ($j = 1, 2$) is not given here. To complement the SIE system (4), (5) it is necessary to add the conditions of ring equilibrium and unambiguity of $\Theta_0(t)$

$$\int_{L_0} \exp(i\psi(t)) \{ p_1(\tau) + p_2(\tau) \} ds = X_0 + iY_0, \tag{6}$$

$$\int_{L_0} \left[\frac{a^2 - b^2}{2ab} \sin(2v) p_1(\tau) + ab p_{23}(\tau) \right] dv = M_0,$$

$$\tau = a \cos v + ib \sin v, \tag{7}$$

$$\sum_{j=1}^2 \int_{L_0} T_j(a, t) p_j(\tau) ds + \sum_{i=1}^3 g_i V(a) = 0, \tag{8}$$

as well as unambiguity condition for displacements during bypassing each crack on L_j

$$\int_{L_j} \omega(\tau) d\tau = 0 \quad (j = \overline{1, k}); \tag{9}$$

$X_0 + iY_0$, M_0 is the principal vector and the moment of external forces $p_1^*(\tau)$, $p_2^*(\tau)$, applied to the ring. Thus, the problem has been reduced to the solution of the SIE system (4), (5) in combination with the additional conditions (6)-(9).

Let the strengthening ring have a small width. Then we may neglect the rigidity in bending and assume that the ring works only in tension-compression. From the equilibrium condition of an infinitesimal element of the ring we can express $p_1(t)$ in terms of $p_2(t)$

$$\begin{aligned}
p_1(t) &= \frac{1}{\rho(t)} \left\{ \rho^*(t) p_2^*(t) + \int_0^{s(t)} \left[\frac{\rho^*(\tau)}{\rho(\tau)} p_2^*(\tau) - p_2(\tau) \right] ds + \sigma(0) G(0) \right\}, \\
\rho^*(t) &= \rho(t) - b^*(t).
\end{aligned} \tag{10}$$

Here $\rho(t)$, $b^*(t)$ are the radius of curvature and the width of the strengthening ring at point t ; $\sigma(0)$, $G(0)$ are the tension-compression stress and the area of the ring cross section at point $t = a$.

We obtain SIE on the contour L_0 from the condition of compatibility of the plate and strengthening ring deformations in the direction of the tangent $\bar{\tau}$

$$\begin{aligned}
& \sum_{j=1}^2 \int_{L_0} K_{1j}^*(t, \tau) p_j(\tau) ds + \operatorname{Re} \{ [K_{13}^*(t, \tau) \omega(\tau) + K_{14}^*(t, \tau) \overline{\omega(\tau)}] ds \} + \\
& + \frac{\sigma(0)G(0)}{E_0 G(t)} = c_1^*(t) \quad t \in L_0, \\
& K_{1j}^*(t, \tau) = 2 \operatorname{Re} \left\{ \sum_{v=1}^2 \frac{S_v^*(\alpha)}{\omega_v(\xi_v)} \cdot \frac{P_{vj}(\xi \eta)}{\xi - \eta} \right\} - \frac{(j-1)}{E_0 G(t)} \delta(t, \tau); \\
& K_{13}^*(t, \tau) = \sum_{v=1}^2 \frac{S_v^*(\alpha)}{\pi i \omega_v(\xi)} \left\{ \frac{(-1)^{v-1} a^{v-1}}{\eta_v - \xi} \times \right. \\
& \quad \left. d\tau_v/ds - Q_v(\xi, \eta_v) \overline{b(\tau)} d\bar{\tau}_2/ds \right\}, \\
& K_{14}^*(t, \tau) = \sum_{v=1}^2 \frac{S_v^*(\alpha)}{\pi i \omega_v(\xi)} \left\{ R_v(\xi, \eta_v) d\bar{\tau}_1/ds - \right. \\
& \quad \left. - \overline{a(\tau)} Q_v(\xi, \eta_v) d\bar{\tau}_2/ds + \frac{(1-v)b(\tau) d\tau_v/ds}{\eta_v - \xi} \right\}, \\
& c_1^*(t) = -\frac{1}{E_0 G(t)} \int_0^{s(t)} \frac{\rho^*(\tau)}{\rho(\tau)} p_2^*(\tau) ds - \\
& \quad - 2 \operatorname{Re} \left\{ \sum_{v=1}^2 S_v^*(\alpha) \Phi_{v0}(t_v) \right\}, \\
& S_v^*(\alpha) = (p_v \cos \alpha + q_v \sin \alpha) (\cos \alpha + \mu_v \sin \alpha), \\
& \alpha = \psi(t) + \pi/2, \quad \delta(t, \tau) = \begin{cases} 1, & \tau < t \\ 0, & \tau > t \end{cases}
\end{aligned} \tag{11}$$

The SIE system (5), (10), (11) in combination with the additional conditions (7), (9) yields the solution of the problem for the case of a thin strengthening ring.

By the known method [1] we reduce the SIE system (5), (11), combined with (10), to a system of linear equations relative to approximate values of the unknown functions

$$\begin{aligned}
& \omega_1(\tau), \quad \{ \tau = \tau(\beta) | -1 < \beta < 1, \tau \in L \}; \\
& p_2(\tau), \quad \{ \tau = a \cos v + i b \sin v, \quad 0 \leq v < 2\pi, \tau \in L \}
\end{aligned}$$

in the nodes

$$\beta_s = \cos \left(\frac{2s-1}{2N} \pi \right), \quad (s = \overline{1, N}), \quad v_j = \frac{2j-1}{2N} \pi, \quad (j = \overline{1, 2N})$$

respectively. Theoretical estimates for the present numerical method convergence are given in [1].

Let us list some results of calculations for orthotropic and isotropic materials of the plate (I is glass epoxide, $E_1 = 5.384 \cdot 10^4$ MPa; $E_1/E_2 = 3$; $E_1/G_{12} = 6.24$; $\nu_1 = 0.25$; II is broken epoxide, $E_1 = 27.61 \cdot 10^4$ MPa; $E_1/E_2 = 10$, $E_1/G_{12} = 26.68$; $\nu_1 = 0.25$; III is graphite epoxide, $E_1 = 27.61 \cdot 10^4$ MPa, $E_1/E_2 = 25$; $E_1/G_{12} = 50$, $\nu_1 = 0.25$; IV is isotropic material, $E_1 = E_2 = 27.61 \cdot 10^4$ MPa, $\nu = 0.33$) with a round hole of radius a , strengthened along the contour of the hole by a closed momentless ring with a relative rigidity $u = E_0 G_0 / E_1 a h$ (Fig. 2)) (E_1 is the Young modulus along the principal direction of anisotropy, which makes an angle with the x axis; G_0 is the constant area of the ring cross section). At infinity the plate is extended by the forces $\sigma_x^{(\infty)} = \sigma$.

Figure 2 demonstrates the dependence between the stress concentration $\sigma_p(\pi/2)/\sigma$ on the hole contour in the plate and the ring rigidity parameter u , degree of the plate material anisotropy, and an angle of anisotropy φ (solid lines correspond to the case $\varphi = 0$ and dotted lines, to $\varphi = \pi/2$). For isotropic plate material (curve 4) the calculation results almost coincided with the data of [8].

Figure 3 illustrates the distribution stresses σ_p/σ , σ_t/σ on the contour of the strengthened round hole in the plate for different plate materials (I, II, and III), and angles $\varphi = 0$; $\pi/2$, $u = 0.1$ at uniaxial extension at infinity $\sigma_x^{(\infty)} = \sigma$.

Figure 4 shows the dependences of the stress intensity factors (SIF) of the break

$$k_1(-1) = K_1/\sigma\sqrt{\pi l},$$

where

$$K_1 = \lim_{x \rightarrow \Delta - l} \sigma_y(x) \sqrt{2\pi(\Delta - l - x)}$$

in the apex of a crack of length $2l$ neighboring to the strengthened round hole ($a = b$) on the parameter $\eta = l/(\Delta - a)$ at $\Delta/a = 2$; $u = 0.2; 0.1; 0.05; 0$ (curves 1-4), $\varphi = 0, \pi/2$ (solid and dotted lines). The value of $u = 0$ corresponds to the case of an unstrengthened hole. The values of SIF for the crack apices neighboring ($k_1(-1)$) to and distant ($k_1(1)$) from the hole at $\eta = 0.1; 0.5; 0.9$; $\varphi = 0$; $u = \infty$ (absolutely rigid ring) and the number of division nodes of the crack contour $N = 10; 20; 30$ (the hole contour is divided into $2N$ nodes) are given in the Table 1. As is seen from the table the convergence of the results are somewhat deteriorated when the crack approaches the hole.

The effect of the hole ellipticity b/a on SIF $k_1(-1)$ at $\eta = 0.7$; $\Delta/a = 2$; $u = 0.2; 0.1; 0.05; 0$ (curves 1-4), $\varphi = 0; \pi/2$ (solid and dotted lines) is shown in Fig. 5. Note that for $\Delta/b, l/b < 1$ the calculation results are in good agreement with the data obtained for the case of half-plane, strengthened along the edge by an infinite strengthening rib and weakened by a buried crack [4].

The results given here demonstrate that the parameters of the plate material anisotropy, strengthening ring rigidity, and the hole shape affect essentially stress concentration in the plate and the value of SIF at the apices of the crack. The effect of the plate material anisotropy is the more significant the higher the rigidity of the strengthening ring. When an elliptic hole is "compressed" in the direction perpendicular to the crack, the dependence of SIF in the cracks on anisotropy parameters is reduced. The calculation results given in the table demonstrate a good convergence of the numerical method of solving the problem. To check the numerical results an experiment was carried out on determining SIF in the crack near an edged hole by a polarization-optical method [2]. In most cases disagreement between theoretical and experimental data does not exceed 5%.

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