

MICROCRACKS WITH PREFRACTURE ZONES IN THE VICINITY OF A RIGID INCLUSION

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We establish general relationships for the analysis of local stress fields induced by small cracklike defects and inclusions in the vicinity of a macroscopic rigid inclusion, determine the asymptotic solutions of the corresponding singular integral equations, and construct an analog of the δ_k -model for small cracks in the vicinity of a macroscopic inclusion or a crack. We also present formulas for the numerical evaluation of the parameters of the δ_k -model for a microcrack perpendicular to the macroscopic inclusion.

The processes of initiation and propagation of cracks in materials are, as a rule, induced by technological and in-service defects. The cases of two or more interacting unequal cracks [1–3] and cracks interacting with rigid inclusions [4, 5] were studied in detail by the methods of the theories of functions of complex variables and singular integral equations in the two-dimensional formulation. However, the major part of results known from the literature can be described as asymptotic solutions of systems of singular integral equations obtained for sufficiently remote defects and the available numerical data accumulated for close inclusions and cracks of different lengths cannot be used instead of the analytic formulas required for the construction of criteria used to monitor the processes of initiation and propagation of fracture near macroscopic defects. In [6], the singular equations suggested in [2, 3] were solved by the method of two approximations for the case where one crack is much longer than the other cracks located in its vicinity. The stress intensity factors near a macroscopic rigid inclusion in the material surrounded by small cracks and inclusions were estimated in [5]. In this monograph, one can also find numerical values of some other parameters of fracture mechanics.

In the present work, we obtain asymptotic solutions of singular integral equations for a large rigid inclusion interacting with short rigid inclusions and small cracks with prefracture zones at their tips. These solutions are then generalized to an analog of the δ_k -model [1] for a system of small cracks interacting with a macroscopic defect.

Statement of the Problem. System of Basic Integral Equations

We consider an infinite elastic homogeneous plane containing a large defect (a perfectly rigid inclusion or a crack). A coordinate system xOy is related to the contour L_0 of this inclusion so that the latter occupies a segment $|x| < 2l_0$. This inclusion is surrounded by N arbitrarily located small curvilinear defects (m cracks and $N - m$ rigid inclusions) with contours L_n ($n = 1, 2, 3, \dots, N$). Each of these small defects is related to a local coordinate system $x_nO_ny_n$ whose origin coincides with the center of the defect $z_n^0 = x_n^0 + iy_n^0$ and the O_nx_n -axis is directed along its tangent. We assume that the defects are disjoint (see Fig. 1) and the crack contours are loaded by forces that are not self-balanced, namely,

$$N_n^\pm(t_n) + iT_n^\pm(t_n) = p_n(t_n) \pm q_n(t_n), \quad n = 0, 1, 2, 3, \dots, m. \quad (1)$$

Moreover, the derivatives of displacements

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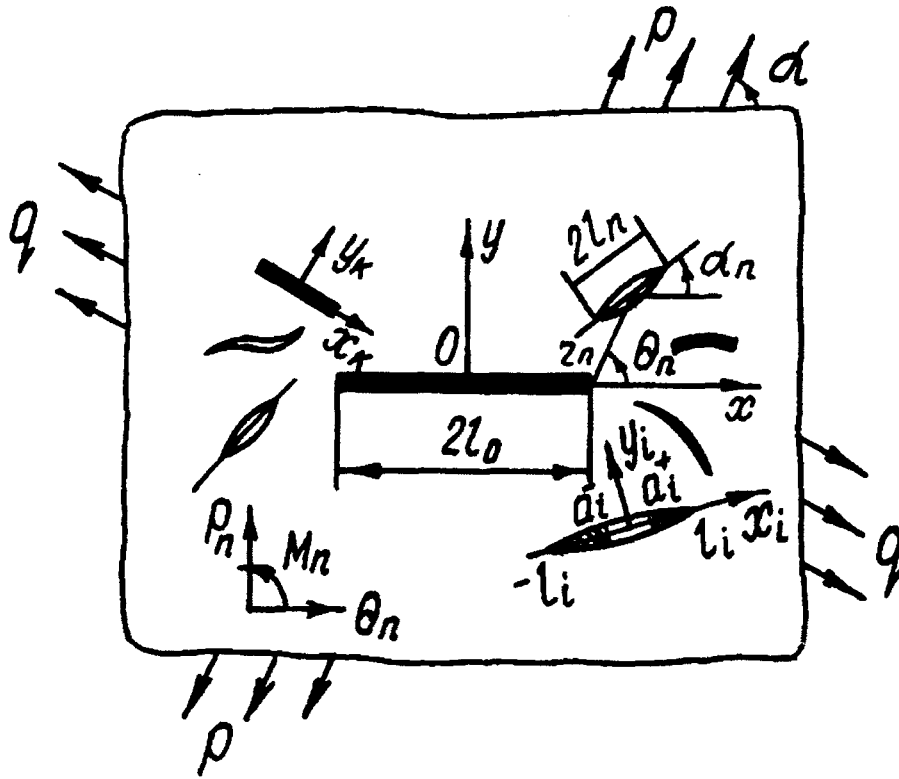


Fig. 1. System of small cracks and inclusions in the vicinity of a macroscopic defect.

$$2G \frac{d}{dt_n} (u_n^\pm - iv_n^\pm) = f_n'(t_n) \pm g_n'(t_n), \quad n = 0, m+1, m+2, \dots, N. \quad (2)$$

are defined on the contours of the rigid inclusions. At infinitely remote points of the plane, stresses and rotations are absent. In relations (1) and (2), the upper and lower signs correspond to the limiting values of stresses and displacements on the opposite lips of the defects, t_n is an arc coordinate on the contour of a crack (inclusion), and G is the shear modulus of the material. For each defect, the projections X_n and Y_n of the principal vector of forces onto the $O_n x_n$ and $O_n y_n$ axes are known. It is necessary to determine the stress-strain state of the elastic plane with this system of defects, the stress intensity factors in the vicinity of the tips of stress concentrators, and the sizes of prefracture zones near the tips of small cracks.

If we now introduce the following notation common for the first and second main problems of the theory of elasticity:

$$\{G_n(\rho_n, t_n), F_n(\rho_n, t_n)\} = \begin{cases} \{-2q_n(t_n), -2p_n(t_n)\}, & \rho_n^* = -1, \\ \{2g_n'(t_n), 2f_n'(t_n)\}, & \rho_n^* = \chi, \end{cases}$$

then the system of singular integral equations guaranteeing the validity of the boundary conditions (1) and (2) takes the form

$$\sum_{k=0}^N \int_{L_k} [K_{nk}(t_k, t'_n; \rho_n^*, \rho_k^*) Q_k(t_k, \rho_k^*, \rho_k) dT_k + L_{nk}(t_k, t'_n) \overline{Q_k(t_k, \rho_k^*, \rho_k)} d\bar{T}_k + M_{nk}(t_k, t'_k) G_k(\rho_k^*, t_k) dT_k] = \pi F_n(\rho_n^*, t'_n), \quad t'_n \in L_n, \quad n = 0, 1, 2, 3, \dots, N, \quad (3)$$

where

$$Q_k(t_k, \rho_k^*, \rho_k) = \frac{G_k(\rho_k, t_k) - G_k(\rho_k^*, t_k)}{\rho_k - \rho_k^*},$$

$G_k(\rho_k, t_k)$ are unknown functions of the jumps of displacements (for cracks) or of the jumps of stresses (for inclusions), $K_{nk}(t_k, t'_n; \rho_k^*, \rho_k^*)$, $L_{nk}(t_k, t'_n)$, and $M_{nk}(t_k, t'_n)$ are the kernels of this system of equations given by the formulas

$$K_{nk}(t_k, t'_n; \rho_k^*, \rho_k^*) = \frac{e^{i\alpha_k}}{i} \left[\frac{\rho_n^* + \rho_k^*}{T_k - T_n'} + \frac{d}{dT_n'} \left(\frac{T_k - T_n'}{T_k - T_n'} \right)^{\rho_k^*} \right],$$

$$L_{nk}(t_k, t'_n) = i e^{i\alpha_k} \frac{d}{dT_n'} \left(\frac{T_k - T_n'}{T_k - T_n'} \right), \quad M_{nk}(t_k, t'_n) = \frac{i e^{i\alpha_k}}{T_k - T_n'} \frac{dT_n'}{dT_n}, \quad (4)$$

$T_k = t_k \exp(i\alpha_k) + z_k^0$, α_k is the orientation of the k th local coordinate system with respect to the Ox -axis, the parameter $\rho_k^*(\rho_k)$ is equal to -1 (χ) for the case of a crack and to $\chi(-1)$ for a rigid inclusion, $\chi = (3 - \mu)/(1 + \mu)$ for the plane stressed state, $\chi = 3 - 4\mu$ for the conditions of plane deformation, and μ is Poisson's ratio.

The solutions of system (3) must satisfy the following generalized conditions:

$$\int_{L_n} G_n(\rho_n, t_n) dT_n = i \frac{1 + \rho_n^*}{\rho_n - \rho_n^*} (X_n + iY_n), \quad n = 0, 1, 2, 3, \dots, N, \quad (5)$$

guaranteeing the uniqueness of displacements in the case of cracks ($\rho_n^* = -1$, $\rho_n = \chi$) and determining the projections X_n and Y_n of the principal vector of forces in tracing each contour L_n in the case of inclusions ($\rho_n^* = \chi$, $\rho_n = -1$). If conditions (5) are satisfied, then the integral equations (3) are uniquely solvable in the class of functions unbounded at the ends of the defects.

Without loss of generality, we solve the problem under consideration for the case of rectilinear crack-like defects. We pass to dimensionless variables $t_k = l_k \tau$ and $t'_n = l_n \eta$, where l_k and l_n are the half lengths of linear defects, and regularize the system of singular integral equations (3) according to [8]. As a result, in the case where the contours of the defects are subjected to the action of self-balanced forces (i.e., for $G_i(\rho_i^*, t_i) \equiv 0$), we get

$$G_0(\rho_0; l_0 \eta) = \frac{i(\rho_0 - \rho_0^*)}{2\pi \rho_0^* l_0 \sqrt{1 - \eta^2}} \left\{ -l_0 \int_{-1}^1 \frac{\sqrt{1 - \tau^2} F_0(\rho_0^*; l_0 \tau)}{\tau - \eta} d\tau \right.$$

$$+ \sum_{k \neq 0} \frac{l_k}{\rho_k - \rho_{k-1}^*} \int_{-1}^1 [G_k(\rho_k; l_k \tau) M_{0k}(l_k \tau, l_0 \eta; \rho_0^*, \rho_k^*) + \overline{G_k(\rho_k; l_k \tau)} N_{0k}(l_k \tau, l_0 \eta)] d\tau \left. \right\}, \quad (6)$$

$$|\tau| \leq 1, \quad |\eta| \leq 1,$$

$$G_n(\rho_n; l_n \eta) = \frac{i(\rho_n - \rho_n^*)}{2\pi \rho_n^* l_n \sqrt{1 - \eta^2}} \left\{ -l_n \int_{-1}^1 \frac{\sqrt{1 - \tau^2} F_n(\rho_n^*; l_n \tau)}{\tau - \eta} d\tau \right.$$

$$\begin{aligned}
& + \frac{l_0}{\rho_0 - \rho_0^*} \int_{-1}^1 [G_0(\rho_k; l_0 \tau) M_{n0}(l_0 \tau, l_n \eta; \rho_n^*, \rho_0^*) + \overline{G_0(\rho_0; l_0 \tau)} N_{n0}(l_0 \tau, l_n \eta)] d\tau \\
& + \sum_{\substack{k=1 \\ k \neq n}}^N \frac{l_k}{\rho_k - \rho_{k-1}^*} \int_{-1}^1 [G_k(\rho_k; l_k \tau) M_{nk}(l_k \tau, l_n \eta; \rho_n^*, \rho_k^*) + \overline{G_k(\rho_k; l_k \tau)} N_{nk}(l_k \tau, l_n \eta)] d\tau \Big\}, \quad |\eta| \leq 1, \quad (7)
\end{aligned}$$

$$\begin{aligned}
M_{nk}(l_k \tau, l_n \eta; \rho_n^*, \rho_k^*) &= -\frac{e^{i\alpha_{kn}}}{\pi i} l_n \int_{-1}^1 \frac{\sqrt{1-\xi^2}}{\xi - \eta} \left(\frac{\rho_k^*}{l_n \xi - T_{nk}} + \frac{\rho_n^*}{l_n \xi - \bar{T}_{nk}} \right) d\xi, \\
N_{nk}(l_k \tau, l_n \eta) &= -\frac{(T_{nk} - \bar{T}_{nk}) e^{i\alpha_{nk}}}{\pi i} l_n \int_{-1}^1 \frac{\sqrt{1-\xi^2}}{(\xi - \eta)(l_n \xi - T_{nk})^2} d\xi, \quad (8)
\end{aligned}$$

where

$$\alpha_{kn} = \alpha_k - \alpha_n, \quad T_{nk} = e^{i\alpha_{nk}} \left[\tau l_k - d_{nk} e^{i(\alpha_k - \beta_{nk})} \right],$$

d_{nk} is the distance between the centers of the n th and k th defects, i.e., $z_n^0 - z_k^0 = d_{nk} e^{i\beta_{nk}}$, and β_{nk} is the angle determining their mutual orientation. The regularized representation (6), (7) of the system of singular integral equations (3) is convenient for the construction of its asymptotic solutions.

Asymptotic Solution of the Basic System of Singular Integral Equations

We assume that all small defects located in the vicinity of the large defect have equal lengths $l_k = l$ ($k = 1, 2, 3, \dots, N$) and construct solutions of the system of equations (6), (7) in the form of series in powers of the small parameter $\lambda = l/l_0$:

$$G_0(\rho_0, l_0 \eta) = \sum_{p=0}^{\infty} G_{0p}(\rho_0, \eta) \lambda^p, \quad (9)$$

$$G_n(\rho_n, l \eta) = \sum_{p=0}^{\infty} G_{np}(\rho_n, \eta) \lambda^p. \quad (10)$$

The expansions of expressions (8) in power series of the parameter λ have the form

$$\begin{aligned}
M_{0k}(l \tau, l_0 \eta; \rho_0^*, \rho_k^*) &= \sum_{p=0}^{\infty} m_{0kp} \lambda^p, & N_{0k}(l \tau, l_0 \eta) &= \sum_{p=0}^{\infty} n_{0kp} \lambda^p, \\
M_{k0}(l_0 \tau, l \eta; \rho_k^*, \rho_0^*) &= \sum_{p=1}^{\infty} m_{k0p} \lambda^p, & N_{k0}(l_0 \tau, l \eta) &= \sum_{p=1}^{\infty} n_{k0p} \lambda^p, \\
M_{nk}(l \tau, l \eta; \rho_n^*, \rho_k^*) &= \sum_{p=1}^{\infty} m_{nkp} \lambda^p, & N_{nk}(l \tau, l \eta) &= \sum_{p=1}^{\infty} n_{nkp} \lambda^p,
\end{aligned} \quad (11)$$

where

$$m_{0kp}(\tau, \eta; \rho_0^*, \rho_k^*) = \frac{ie^{i\alpha_k}}{\pi} \tau^p \left[\rho_0^* e^{ip\alpha_k} \int_{-1}^1 \frac{\sqrt{1-\xi^2} d\xi}{(\xi-\eta)(\xi-u_k^0)^{p+1}} + \rho_k^* e^{ip\alpha_k} \int_{-1}^1 \frac{\sqrt{1-\xi^2} d\xi}{(\xi-\eta)(\xi-\bar{u}_k^0)^{p+1}} \right], \quad (12)$$

$$n_{0kp}(\tau, \eta) = -\frac{2e^{-i\alpha_k} y_k^0}{\pi l_0} \tau^p \int_{-1}^1 \frac{\sqrt{1-\xi^2}}{\xi-\eta} \left[(p+1) \frac{e^{-ip\alpha_k}}{(\xi-u_k^0)^{p+2}} + \frac{l_0 p \sin \alpha_k e^{-i(p-1)\alpha_k}}{y_k^0 (\xi-\bar{u}_k^0)^{p+1}} \right] d\xi,$$

$$m_{k0p}(\tau, \eta; \rho_0^*, \rho_k^*) = \frac{e^{-i\alpha_k}}{i} \left[\frac{\rho_0^*}{(T_{k0} l_0^{-1})^p} + \frac{\rho_k^*}{(\bar{T}_{k0} l_0^{-1})^p} \right] H_{p-1}(\eta), \quad (13)$$

$$n_{k0p}(\tau, \eta) = \frac{e^{i\alpha_k}}{i} p \frac{T_{k0} l_0^{-1} - \bar{T}_{k0} l_0^{-1}}{(T_{k0} l_0^{-1})^{p+1}} H_{p-1}(\eta),$$

$$m_{nkp}(\tau, \eta; \rho_n^*, \rho_k^*) = \frac{e^{i\alpha_{kn}}}{\pi i} \left[\frac{\rho_n^* e^{ip\alpha_n}}{(u_k^0 - u_n^0)^p} \int_{-1}^1 \frac{\sqrt{1-\xi^2} (\xi - \tau e^{-i\alpha_{nk}})^{p-1}}{\xi - \eta} d\xi + \frac{\rho_k^* e^{-ip\alpha_n}}{(\bar{u}_k^0 - \bar{u}_n^0)^p} \int_{-1}^1 \frac{\sqrt{1-\xi^2} (\xi - \tau e^{i\alpha_{nk}})^{p-1}}{\xi - \eta} d\xi \right], \quad (14)$$

$$n_{nkp}(\tau, \eta) = \frac{2e^{i\alpha_{kn}}}{\pi i} \frac{e^{-ip\alpha_k}}{(\bar{u}_k^0 - \bar{u}_n^0)^p} \int_{-1}^1 \frac{\sqrt{1-\xi^2} (\xi - \tau e^{i\alpha_{nk}})^p}{\xi - \eta} d\xi \times \left[\frac{p(\tau e^{i\alpha_{nk}} - \xi)}{e^{i\alpha_n} (\bar{u}_k^0 - \bar{u}_n^0)} \operatorname{Im} \left\{ e^{i\alpha_n} (\bar{u}_n^0 - \bar{u}_k^0) \right\} + i(p-1)\tau \sin \alpha_{nk} \right],$$

and $u_k^0 = z_k^0/l_0$. The values of the singular integral in relations (13)

$$H_p(\eta) = \frac{1}{\pi} \int_{-1}^1 \frac{\tau^p \sqrt{1-\tau^2}}{\tau - \eta} d\tau, \quad |\eta| < 1,$$

are given by the well-known recurrence relations presented in [9].

For the convergence of series (9)–(11), it is necessary that

$$\left| \frac{\lambda}{\tau - u_k^0} \right| < 1,$$

where $|\tau| < 1$.

By substituting relations (9)–(11) in Eqs. (6) and (7) and comparing the coefficients of the same powers of the small parameter λ , we arrive at the following system of recurrence relations for the coefficients of series (9) and (10):

$$G_{00}(\rho_0, \eta) = -\frac{i(\rho_0 - \rho_0^*)}{2\pi\rho_0^*\sqrt{1-\eta^2}} \int_{-1}^1 \frac{\sqrt{1-\tau^2} F_0(\rho_0^*, \tau)}{\tau - \eta} d\tau,$$

$$G_{0S}(\rho_0, \eta) = \frac{i(\rho_0 - \rho_0^*)}{2\pi\rho_0^*\sqrt{1-\eta^2}} \sum_{k=1}^N \frac{l_k}{\rho_k - \rho_k^*} \int_{-1}^1 \left\{ \sum_{p=0}^{S-1} [G_{kp}(\rho_k, \tau) m_{0kS-1-p} + \overline{G_{kp}(\rho_k, \tau)} n_{0kS-1-p}] \right\} d\tau, \quad (15)$$

$$G_{n0}(\rho_n, \eta) = \frac{i(\rho_n - \rho_n^*)}{2\pi\rho_n^*\sqrt{1-\eta^2}} \left\{ \int_{-1}^1 \frac{\sqrt{1-\tau^2} F_n(\rho_n^*, \tau)}{\tau - \eta} d\tau + \frac{l_k}{\rho_0 - \rho_0^*} \int_{-1}^1 [G_{00}(\rho_0, \tau) m_{n01} + \overline{G_{00}(\rho_0, \tau)} n_{n01}] d\tau \right\}, \quad (16)$$

$$G_{nS}(\rho_n, \eta) = \frac{i(\rho_n - \rho_n^*)}{2\pi\rho_n^*\sqrt{1-\eta^2}} \left\{ \frac{1}{\rho_0 - \rho_0^*} \int_{-1}^1 \sum_{p=0}^S [G_{0p}(\rho_0, \tau) m_{n0S+1-p} + \overline{G_{0p}(\rho_0, \tau)} n_{n0S+1-p}] d\tau \right. \\ \left. + \sum_{\substack{k=1 \\ k \neq n}}^N \frac{1}{\rho_k - \rho_k^*} \int_{-1}^1 [G_{kp}(\rho_k, \tau) m_{nkS-p} + \overline{G_{kp}(\rho_k, \tau)} n_{nkS-p}] d\tau \right\}.$$

In the general case of biaxial tension of the plate by mutually perpendicular forces with intensities p and q such that the forces p are directed at the right angle to the Ox -axis, it is necessary to assume that [4]

$$F_n(\rho_n^*, t) = 0.5p[(1 - \rho_n^*)(1 + \eta_0) + 2(1 - \eta_0)e^{-2i\alpha_n}] + 4iG\varepsilon_n, \quad n = 0, 1, 2, 3, \dots, N, \quad \eta_0 = \frac{q}{p}, \quad (17)$$

on the contours of the defects, where ε_n is the angle of rotation of the n th rigid inclusion (for cracks, $\varepsilon_n = 0$). By substituting (17) in the recurrence relations (15) and (16), we deduce explicit expressions for the coefficients of series (9) and (10), i.e., construct the required solutions of the basic integral equations.

The indicated type of biaxial tension of the plate at infinity corresponds to the action of a self-balanced load given by (17) upon the contours of the defects, i.e., to the validity of the equalities $G_i(\rho_i^*, t) = 0$, $i = 0, 1, 2, 3, \dots, N$. By using conditions (5), we can now show that

$$\int_{-1}^1 [G_{k0}(\rho_k, \tau) m_{0k0} + \overline{G_{k0}(\rho_k, \tau)} n_{0k0}] d\tau = 0. \quad (18)$$

Similarly,

$$\int_{-1}^1 [G_{k1}(\rho_k, \tau) m_{0k0} + \overline{G_{k1}(\rho_k, \tau)} n_{0k0}] d\tau = 0, \quad (19)$$

$$\int_{-1}^1 [G_{k0}(\rho_k, \tau) m_{nk1} + \overline{G_{k0}(\rho_k, \tau)} n_{nk1}] d\tau = 0. \quad (20)$$

In view of relations (18) and (15), we obtain

$$G_{01}(\rho_0, \eta) = 0 \quad (21)$$

and, according to formulas (15) and (19), the next coefficient of expansion (9) has the form

$$G_{02}(\rho_0, \eta) = \frac{i(\rho_0 - \rho_0^*)}{2\pi\rho_0^*\sqrt{1-\eta^2}} \sum_{k \neq 0} \frac{1}{\rho_k - \rho_k^*} \int_{-1}^1 [G_{k0}(\rho_k, \tau)m_{0k1} + \overline{G_{k0}(\rho_k, \tau)}n_{0k1}] d\tau. \quad (22)$$

Therefore, to compute the coefficient $G_{02}(\rho_0, \eta)$, it suffices to know the quantities $G_{k0}(\rho_k, \eta)$. Note that if the forces are uniformly distributed at infinity, then the quantity $F_k(\rho_k^*, \tau)$ does not depend on τ and we can set

$$F_k(\rho_k^*, \tau) = s_k(\rho_k^*). \quad (23)$$

Substituting equality (23) in relations (16), we get

$$G_{k0}(\rho_k, \eta) = \frac{i(\rho_k - \rho_k^*)}{2\pi\rho_k^*\sqrt{1-\eta^2}} \left\{ \pi s_k(\rho_k^*)\eta + \frac{1}{\rho_0 - \rho_0^*} \int_{-1}^1 [G_{00}(\rho_k, \tau)m_{k01} + \overline{G_{00}(\rho_k, \tau)}n_{k01}] d\tau \right\}. \quad (24)$$

Similarly, by using equalities (20) and (21) and relation (16), we find

$$G_{k1}(\rho_k, \eta) = \frac{i(\rho_k - \rho_k^*)}{2\pi\rho_k^*(\rho_0 - \rho_0^*)\sqrt{1-\eta^2}} \int_{-1}^1 [G_{00}(\rho_0, \tau)m_{k02} + \overline{G_{00}(\rho_0, \tau)}n_{k02}] d\tau. \quad (25)$$

According to formulas (12)–(14), we have the following expressions for the quantities m_{ijs} and n_{ijs} appearing in relations (18), (22), (24), and (25):

$$\begin{aligned} m_{0k0}(\tau, \eta; \rho_0^*, \rho_k^*) &= -\frac{e^{i\alpha_k}}{i} \left[\sqrt{\eta^2 - 1} \left(\frac{\rho_0^*}{\eta - u_k^0} + \frac{\rho_k^*}{\eta - u_k^0} \right) - (\rho_0^* + \rho_k^*) - \rho_0^* \frac{\sqrt{(u_k^0)^2 - 1}}{\eta - u_k^0} - \rho_k^* \frac{\sqrt{(u_k^0)^2 - 1}}{\eta - u_k^0} \right], \\ m_{0k1}(\tau, \eta; \rho_0^*, \rho_k^*) &= i\tau e^{i\alpha_k} \left\{ \rho_0^* e^{i\alpha_k} \left[\frac{\sqrt{\eta^2 - 1}}{(\eta - u_k^0)^2} + \frac{1 - \eta u_k^0}{(\eta - u_k^0)^2 \sqrt{(u_k^0)^2 - 1}} \right] \right. \\ &\quad \left. + \rho_k^* e^{-i\alpha_k} \left[\frac{\sqrt{\eta^2 - 1}}{(\eta - u_k^0)^2} + \frac{1 - \eta \overline{u_k^0}}{(\eta - \overline{u_k^0})^2 \sqrt{(u_k^0)^2 - 1}} \right] \right\}, \\ n_{0k0}(\tau, \eta) &= -\frac{2e^{-i\alpha_k} y_0}{l_0} \left[\frac{\sqrt{\eta^2 - 1}}{(\eta - u_k^0)^2} + \frac{1 - \eta \overline{u_k^0}}{(\eta - \overline{u_k^0})^2 \sqrt{(u_k^0)^2 - 1}} \right], \\ n_{0k1}(\tau, \eta) &= -2\tau e^{-i\alpha_k} \left\{ \frac{y_k^0}{l_0} e^{-i\alpha_k} \left[\frac{2\sqrt{\eta^2 - 1}}{(\eta - u_k^0)^3} + \frac{2\eta(u_k^0)^3 - \eta^2 - 3\overline{u_k^0} + 2}{(u_k^0 - \eta)^3 (u_k^0 - 1)^{3/2}} \right] \right. \\ &\quad \left. + \sin \alpha_k \left[\frac{\sqrt{\eta^2 - 1}}{(\eta - u_k^0)^2} + \frac{1 - \eta \overline{u_k^0}}{(\overline{u_k^0} - \eta)^2 \sqrt{(u_k^0)^2 - 1}} \right] \right\}, \end{aligned} \quad (26)$$

$$\begin{aligned}
m_{k00} &= 0, \quad n_{k00} = 0, \quad m_{k01}(\tau, \eta; \rho_k^*, \rho_0^*) = -\frac{e^{-i\alpha_k}}{i} \left(\frac{\rho_0^*}{T_{k0} l_0^{-1}} + \frac{\rho_k^*}{\bar{T}_{k0} l_0^{-1}} \right) \eta, \\
m_{k02}(\tau, \eta; \rho_k^*, \rho_0^*) &= \frac{e^{-i\alpha_k}}{i} \left(\frac{1}{2} - \eta^2 \right) \left[\frac{\rho_0^*}{(T_{k0} l_0^{-1})^2} + \frac{\rho_k^*}{(\bar{T}_{k0} l_0^{-1})^2} \right], \\
n_{k01}(\tau, \eta) &= \frac{e^{i\alpha_k}}{i} \left(\frac{\bar{T}_{k0}}{T_{k0}} - 1 \right) \frac{\eta}{T_{k0} l_0^{-1}}, \quad n_{k02}(\tau, \eta) = \frac{2e^{i\alpha_k}}{i} \left(1 - \frac{\bar{T}_{k0}}{T_{k0}} \right) \frac{0.5 - \eta^2}{(T_{k0} l_0^{-1})^2}.
\end{aligned} \tag{26}$$

If we now substitute relations (26) in expressions (22), (24), and (25) and compute the corresponding integrals by using representations (9) and (10) with an accuracy of up to the terms of the same order of smallness as λ^3 , then we obtain the required solutions of the basic singular equations (3). These solutions can be used to determine the stress intensity factors near a macroscopic defect interacting with small defects (cracks and rigid inclusions).

Stress Intensity Factors Near a Macroscopic Inclusion and Microdefects in Its Vicinity

To determine the stress intensity factors near interacting cracks and inclusions and in the vicinity of a macroscopic defect, we use the following formula [5]:

$$K_{In}^{\pm} - iK_{IIIn}^{\pm} = \pm \frac{\rho_n^*}{i(\rho_n - \rho_n^*)} \sqrt{\pi l_n} \lim_{\eta \rightarrow \pm 1} \sqrt{1 - \eta^2} G_n(\rho_n, l_n \eta), \tag{27}$$

where the upper and lower signs correspond to the right and left tips of the defect, respectively. The moments of forces acting against the rotations of rigid inclusions can be found by equating the angle of rotation of the rigid inclusion to zero. Similarly, the angles of rotation are determined by equating the principal moment of forces acting upon each inclusion to zero. Thus, in particular, we have [5]

$$M_n = \frac{\rho_n^*}{\rho_n - \rho_n^*} l_n^2 \operatorname{Re} \left\{ \int_{-1}^1 \eta \left[\frac{G_n(\rho_n, l_n \eta)}{\rho_n^*} - \overline{G_n(\rho_n, l_n \eta)} \right] d\eta \right\}. \tag{28}$$

Substituting the explicit expressions (9) and (10) accurate to within the terms of the third order of smallness with respect to λ in relations (27), we arrive at the following expressions for the stress intensity factors:

$$\begin{aligned}
K_{I0}^{\pm} - iK_{II0}^{\pm} &= 0.5\sqrt{\pi l_0} \left\{ s_0(\rho_0^*) + \frac{\lambda^2}{8} \sum_{\substack{k=1, \\ k \neq 0}}^N \left\{ A_k(\rho_k^*, \rho_0^*) \left[\frac{\rho_0^* e^{2i\alpha_k}}{(\pm 1 - u_k^0) \sqrt{(u_k^0)^2 - 1}} + \frac{\rho_k^*}{(\pm 1 - \bar{u}_k^0) \sqrt{(\bar{u}_k^0)^2 - 1}} \right] \right. \right. \\
&\quad \left. \left. + \overline{A_k(\rho_k^*, \rho_k^*)} \left[\frac{(\bar{u}_k^0 - u_k^0)(2\bar{u}_k^0 \pm 1) e^{-2i\alpha_k}}{(\pm 1 - \bar{u}_k^0) ((u_k^0)^2 - 1)^{3/2}} + \frac{1 - e^{-2i\alpha_k}}{(\pm 1 - \bar{u}_k^0) \sqrt{(\bar{u}_k^0)^2 - 1}} \right] \right\} \right\} + O(\lambda^3), \tag{29}
\end{aligned}$$

where

$$\begin{aligned}
A_k(\rho_k^*, \rho_0^*) = & s(\rho_0^*) \left[\frac{e^{-2i\alpha_k}}{\rho_k^*} \left(1 - \frac{\overline{u_k^0}}{\sqrt{(\overline{u_k^0})^2 - 1}} \right) + \frac{1}{\rho_0^*} \left(1 - \frac{u_k^0}{\sqrt{(u_k^0)^2 - 1}} \right) \right] \\
& + \overline{s(\rho_0^*)} \frac{1}{\rho_k^* \rho_0^*} \left[(e^{-2i\alpha_k} - 1) \left(1 - \frac{\overline{u_k^0}}{\sqrt{(\overline{u_k^0})^2 - 1}} \right) - e^{-2i\alpha_k} \frac{u_k^0 - \overline{u_k^0}}{(\sqrt{(\overline{u_k^0})^2 - 1})^3} \right] - \frac{2}{\rho_k^*} s(\rho_k^*), \quad (30)
\end{aligned}$$

and

$$K_{I k}^{\pm} - i K_{II k}^{\pm} = -0.25 \sqrt{\pi l_k} \left[\rho_k^* A_k(\rho_k^*, \rho_0^*) \pm 0.5 \lambda B_k(\rho_k^*, \rho_0^*) \right] + O(\lambda^2), \quad (31)$$

where

$$\begin{aligned}
B_k(\rho_k^*, \rho_0^*) = & e^{-i\alpha_k} \left\{ s_0(\rho_0^*) \left[\frac{e^{-2i\alpha_k}}{(\sqrt{(\overline{u_k^0})^2 - 1})^3} + \frac{\rho_k^*}{\rho_0^*} \frac{e^{2i\alpha_k}}{(\sqrt{(u_k^0)^2 - 1})^3} \right] \right. \\
& \left. - \frac{2 \overline{s(\rho_0^*)}}{\rho_0^* (\sqrt{(\overline{u_k^0})^2 - 1})^3} \left[1 - e^{-2i\alpha_k} - \frac{3(u_k^0 - \overline{u_k^0}) e^{-2i\alpha_k}}{2((\overline{u_k^0})^2 - 1)} \right] \right\}. \quad (32)
\end{aligned}$$

Similar relations can be deduced on the basis of formulas (27).

By using relations (29)–(32) and taking into account the terms of the same order of smallness as λ^2 , we can now compute the stress intensity factors for all orientations of a microdefect near the tip of the macroscopic defect. A large collection of numerical data of this sort can be found in [5].

Analytic Relations of the δ_k -Model of Small Cracks Initiated in the Vicinity of a Macroscopic Defect

The applicability of the asymptotic method is not restricted to the case of brittle materials. Thus, we consider the following function $F_k(\rho_k^*, t_k)$ specifying the boundary conditions on the contours of small defects:

$$F_k(\rho_k^*, t_k) = \begin{cases} s_k(\rho_k^*), & a_k^- < t < a_k^+, \\ s_k(\rho_k^*) - 2(\sigma_0 + i\tau_0), & -l_k < l < a_k^-, a_k^+ < t < l_k, \end{cases} \quad (33)$$

where $a_k^{\pm}$ correspond to the inner boundaries of the prefracture zones, the quantities $s_k(\rho_k^*)$ are determined from relations (17) and (23), and σ_0 and τ_0 are, respectively, the tensile and shear strengths of the opposite lips of small cracks.

If we now substitute the loading function $F_k(\rho_k^*, t_k)$ in relations (9) and (16), then we get

$$G_k(\rho_k, l\eta) = \frac{i\eta(\rho_k - \rho_k^*)}{4\rho_k^* \sqrt{1 - \eta^2}} \left\{ \rho_k^* A_k(\rho_k^*, \rho_0^*) + \frac{4(\sigma_0 + i\tau_0)}{\pi} \left\{ \frac{\sqrt{1 - \eta^2}}{2\eta} [\Gamma(1, \eta, a_k^+) - \Gamma(1, \eta, a_k^*)] \right. \right.$$

$$+ [\pi - (\arcsin a_k^+ - \arcsin a_k^-)] - \frac{1}{\eta} \left(\sqrt{1 - (a_k^+)^2} - \sqrt{1 - (a_k^-)^2} \right) \left\{ + \lambda^2 \frac{\eta^2 - 0.5}{\eta} B_k(\rho_k^*, \rho_0^*) \right\} + O(\lambda^2). \quad (34)$$

After laborious calculations, we arrive at the following expressions for the jumps of stresses on the macroscopic inclusion:

$$\begin{aligned} G_0(\rho_0, l\eta) = & \frac{i\eta(\rho_0 - \rho_0^*)}{2\rho_0^* \sqrt{1 - \eta^2}} \left\{ s(\rho_0^*) + \frac{\lambda^2}{8\eta} \sum_{k=1}^N \left\{ \left[A_k(\rho_k^*, \rho_0^*) - \frac{2(\sigma_0 + i\tau_0)}{\rho_0^*} \left[\pi - (\arcsin a_k^+ - \arcsin a_k^-) \right. \right. \right. \right. \\ & \left. \left. \left. - \left(a_k^+ \sqrt{1 - (a_k^+)^2} - a_k^- \sqrt{1 - (a_k^-)^2} \right) \right] \right\} \left\{ \rho_0^* e^{2i\alpha_k} \left[\frac{\sqrt{\eta^2 - 1}}{(\eta - u_k^0)^2} + \frac{1 - \eta \overline{u_k^0}}{(\eta - u_k^0)^2 \sqrt{(u_k^0)^2 - 1}} \right] \right. \right. \\ & \left. \left. + \rho_k^* \left[\frac{\sqrt{\eta^2 - 1}}{(\eta - \overline{u_k^0})^2} - \frac{1 - \eta \overline{u_k^0}}{(\eta - \overline{u_k^0})^2 \sqrt{(u_k^0)^2 - 1}} \right] \right\} + \left\{ A_k(\rho_k^*, \rho_0^*) + \frac{2(\sigma_0 - i\tau_0)}{\rho_k^*} \left[\pi - (\arcsin a_k^+ \right. \right. \right. \\ & \left. \left. \left. - \arcsin a_k^-) - \left(a_k^+ \sqrt{1 - (a_k^+)^2} - a_k^- \sqrt{1 - (a_k^-)^2} \right) \right] \right\} \left\{ (\overline{u_k^0} - u_k^0) e^{-2i\alpha_k} \left[\frac{2\sqrt{\eta^2 - 1}}{(\eta - \overline{u_k^0})^3} \right. \right. \right. \\ & \left. \left. \left. + \frac{2\eta(\overline{u_k^0})^3 - \eta^2 - 3(\overline{u_k^0})^2 + 2}{(\overline{u_k^0} - \eta)^3 (\overline{u_k^0} - 1)^{3/2}} \right] + (e^{-2i\alpha_k} - 1) \left[\frac{\sqrt{\eta^2 - 1}}{(\eta - \overline{u_k^0})^3} \right. \right. \right. \\ & \left. \left. \left. + \frac{1 - \eta \overline{u_k^0}}{(\overline{u_k^0} - \eta)^2 \sqrt{(u_k^0)^2 - 1}} \right] + O(\lambda^3) \right\} \right\}. \quad (35) \end{aligned}$$

In relations (34) and (35), we have $u_k^0 = z_k^0/l_0$, the quantities $A_k(\rho_k^*, \rho_0^*)$ and $B_k(\rho_k^*, \rho_0^*)$ are given by expressions (30) and (32), and

$$\Gamma(1, \eta, \xi) = \ln \frac{1 - \eta\xi - \sqrt{(1 - \eta^2)(1 - \xi^2)}}{1 - \eta\xi + \sqrt{(1 - \eta^2)(1 - \xi^2)}}.$$

If we now formally substitute the right-hand-sides of relations (34) and (35) in (27), then we get

$$\begin{aligned} K_{In}^\pm - iK_{II n}^\pm = & -0.25\sqrt{\pi l} \left\{ \rho_n^* A_n(\rho_n^*, \rho_0^*) + \frac{4(\sigma_0 - i\tau_0)}{\pi} \left[\pi - (\arcsin a_n^+ - \arcsin a_n^-) \right. \right. \\ & \left. \left. \pm \left(\sqrt{1 - (a_n^-)^2} - \sqrt{1 - (a_n^+)^2} \right) \right] \pm \frac{\lambda}{2} B_n(\rho_n^*, \rho_0^*) \right\}, \quad (36) \end{aligned}$$

$$\begin{aligned}
K_{I0}^{\pm} - iK_{II0}^{\pm} = & 0.5\sqrt{\pi l_0} \left\{ s_0(\rho_0^*) + \frac{\lambda^2}{8} \sum_{\substack{k=1 \\ k \neq 0}}^N \left\{ \left\{ A_k(\rho_k^*, \rho_0^*) + \frac{2}{\rho_k^*} (\sigma_0 + i\tau_0) \left[\pi - (\arcsin a_k^+ - \arcsin a_k^-) \right. \right. \right. \right. \\
& - \left. \left. \left(a_k^+ \sqrt{1 - (a_k^+)^2} - a_k^- \sqrt{1 - (a_k^-)^2} \right) \right] \right\} \left[\frac{\rho_0^* e^{2i\alpha_k}}{(\pm 1 - u_k^0) \sqrt{(u_k^0)^2 - 1}} + \frac{\rho_k^*}{(\pm 1 - \bar{u}_k^0) \sqrt{(\bar{u}_k^0)^2 - 1}} \right] \\
& - \overline{A_k(\rho_k^*, \rho_0^*)} + 2 \frac{\sigma_0 - i\tau_0}{\rho_k^*} \left[\pi - (\arcsin a_k^+ - \arcsin a_k^-) \right. \\
& - \left. \left. \left(a_k^+ \sqrt{1 - (a_k^+)^2} - a_k^- \sqrt{1 - (a_k^-)^2} \right) \right] \right\} \left[\frac{(\bar{u}_k^0 - u_k^0)(2\bar{u}_k^0 \pm 1)e^{-2i\alpha_k}}{(\pm 1 - \bar{u}_k^0)((\bar{u}_k^0)^2 - 1)^{3/2}} \right. \\
& \left. \left. + \frac{1 - e^{-2i\alpha_k}}{(\pm 1 - \bar{u}_k^0) \sqrt{(\bar{u}_k^0)^2 - 1}} \right] \right\} + O(\lambda^3). \tag{37}
\end{aligned}$$

These expressions enable us to detect the initiation of small cracks in the vicinity of the tips of a macroscopic inclusion with the indicated accuracy. In view of the fact that actual physical microcracks cannot induce infinite stresses in the vicinity of their tips, the equality

$$K_{In}^{\pm} - iK_{IIIn}^{\pm} = 0, \tag{38}$$

by analogy with the δ_k -model [1], yields the values of the lengths a_k^+ and a_k^- of prefraction zones for microcracks located in the vicinity of the macroscopic inclusion. Moreover, by using relations (37) and setting $K_{I0} = K_0 = \text{const}$ [5], we can determine the critical loads corresponding to the appearance of a microcrack.

Evaluation of the Critical Parameters of a Microcrack in the Vicinity of the Tips of a Macroscopic Inclusion

As an example of realization of the procedure described above, we consider the diagram displayed in Fig. 2. According to relations (36) and (37), in view of the fact that the tips of the microcrack are symmetric about the axis of the macroscopic inclusion, we arrive at the following expressions for the stress intensity factors (valid up to the terms of the same order of smallness as λ^2):

$$K_{II}^{\pm} = 0.25\sqrt{\pi l} \left\{ p \frac{9 - \chi^2}{2\chi} \left(1 - \frac{u_1^0}{\sqrt{u_1^2 - 1}} \right) - \frac{8\sigma_0}{\pi} \arccos a \right\}, \tag{39}$$

$$K_{III}^{\pm} = 0.25\sqrt{\pi l} \left\{ \frac{8\tau_0}{\pi} \arccos a \pm \frac{\lambda p(9 - \chi^2)}{4\chi[(u_1^0)^2 - 1]^{3/2}} \right\},$$

$$K_{I0}^{\pm} = 0.25\sqrt{\pi l_0} p(\chi - 3), \quad \text{and} \quad K_{II0}^{\pm} = 0, \tag{40}$$

where

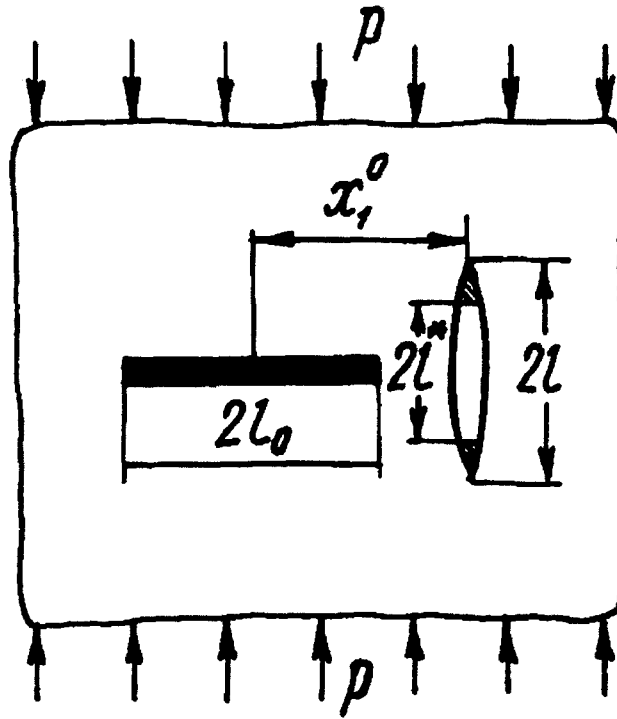


Fig. 2. Macroscopic inclusion with small perpendicular crack.

$$a = a_1^+ = a_1^-, \quad u_1^0 = \frac{x_1^0}{l_0},$$

l_0 is the half length of the macroscopic inclusion, and x_1^0 is the coordinate of the center of the microcrack on the continuation of the stress concentrator. As a result, by using relations (38) and (39), after simple calculations, we can find the total half length of the microcrack and prefraction zones formed at its tips

$$l = l_0 \frac{2\tau_0}{\sigma_0} \frac{(u_1^0)^2 - 1}{(u_1^0)^2 + \sqrt{(u_1^0)^2 - 1}},$$

the half length of the initiated microcrack

$$l^* = l \cos \left[\frac{\pi p_*(9 - \chi^2)}{16\sigma_0 \chi} \left(\frac{u_1^0}{\sqrt{(u_1^0)^2 - 1}} - 1 \right) \right],$$

and the length of the prefraction zone formed at its tip

$$\Delta = 2l \sin^2 \left[\frac{\pi p_*(9 - \chi^2)}{32\sigma_0 \chi} \left(\frac{u_1^0}{\sqrt{(u_1^0)^2 - 1}} - 1 \right) \right].$$

The values of the critical loads p_* corresponding to the appearance of a microcrack can be determined from relations (40) by analyzing either the balance of forces or energy balance [5].

On the basis of the general relations (36) and (37), one can easily establish similar formulas for the case where a microcrack is collinear to the macroscopic inclusion. The maximum possible half length of the initiated collinear microcrack is equal to

$$l^* = l \cos \frac{\pi p_*}{2\sigma_0} \left[1 + \frac{(3-\chi)(\chi-1)}{8\chi} \left(\frac{u_1^0}{\sqrt{(u_1^0)^2 - 1}} - 1 \right) \right].$$

By passing to the limit in this expression as $u_1^0 \rightarrow \infty$, we arrive at the well-known relation of the δ_k -model presented in [1]. This proves the validity of the general results obtained in the present work.

Parallel with the procedures outlined above, it is reasonable to consider some other schemes necessary for engineering calculations [5]. One should especially mention the cases of a δ_k -slot parallel to the macroscopic inclusion and of two slots collinear with this inclusion. In these cases, the lengths of prefracture zones formed at the tips of the microcracks are unequal. Therefore, according to the results of their quantitative evaluation, it is possible to determine the degree of influence of the rigid inclusion on the lengths of prefracture zones located on the continuations of microcracks appearing in the material. The application of the relevant asymptotic relations and basic criteria used to characterize the strength of materials enables one to give an adequate description of the initiation of small cracks near a rigid inclusion.

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