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INTERACTION BETWEEN A BONDED STIFFENER AND A PLATE FASHIONED FROM A COMPOSITE MATERIAL

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Problems involving the computation of plates strengthened by stiffeners assume major significance in the stage of design and subsequent analysis of possible failure modes of the design elements. A review of literature in this direction was conducted, for example, in [1, 2]. Use of standard fasteners does not provide the required strength for plates of composite materials. Glued joints in the stiffener and plate have come into increasingly widespread use in aircraft and space designs.

The problem of the interaction between a stiffener and a semiinfinite elastic anisotropic plate on the assumption that a bonding layer of constant thickness exists between the stiffener and plate is discussed below. The problem reduces to a Fredholm equation of the second kind in terms of unknown contact interaction forces. The character of the behavior of the contact forces is determined. A method is proposed for numerical solution of the integral equation, and results of the computations are presented.

1. Let us examine a semiinfinite elastic plate of constant thickness h_1 , which is fashioned from an anisotropic material having one plane of elastic symmetry parallel to the median plane of the plate $D = \{x \geq 0; |y| < \infty\}$. The plate is strengthened along line $L = \{z = H + e^{i\vartheta}u | 0 < u < l\}$ by an elastic stiffener. The latter is loaded on the ends $u = 0$ and $u = l$ by forces $P_1 e^{i(\vartheta+\pi)}$, $P_2 e^{i\vartheta}$. The plate is tensioned by uniform forces to infinity. We will assume that the stiffener and plate are joined by an elastic bonding layer, which functions only in shear. The splice is more flexible than the stiffener and plate.

Let us investigate the law governing the distribution of forces in the stiffener and determine the running contact forces q that are imparted from the stiffener to the plate (the positive direction for q coincides with the direction of the vector $e^{i(\vartheta+\pi)}$).

Let the bonding layer have a small thickness $\Delta = \text{const}$ with a shear modulus G . The shear strain γ and shear stress τ can be considered uniformly distributed over the cross section of the splice, and can be assumed to just functions of the longitudinal coordinate u .

We will consider that the stiffener acts as a one-dimensional continuum, and the plate is in a generalized plane stressed state. This assumption is valid, if the thickness of the plate, and also the cross-sectional dimensions of the bonding layer and stiffener are small as compared with the length l of the bonded segment.

If the displacement along the u axis of the plate is denoted by U_1 , and that of the stiffener by U_2 , the condition of continuity in our assumptions will assume the form

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$$U_1(u) - U_2(u) = q(u) \Delta (bG)^{-1}, \quad (1.1)$$

where b is the width of the joint. The displacement of the plate along the u axis with an accuracy to an arbitrary constant U is [3, 4]

$$U_1(u) = \int_0^l K(u, v) q(v) dv + Eu + U;$$

$$K(u, v) = 2 \operatorname{Re} \left(\sum_{v=1}^2 T_v \{ A_v \ln [M_v(v-u)] + l_v \bar{A}_v \ln [(H + \bar{M}_1 v) - s_v(H + M_v u)] + n_v \bar{A}_2 \ln [(H + \bar{M}_2 v) - t_v(H + M_v u)] \} \right);$$

$$T_v = p_v \cos \vartheta + q_v \sin \vartheta; \quad M_v = \mu_v \sin \vartheta + \cos \vartheta; \quad A_v = \frac{a_v(\vartheta)}{4\pi h_1};$$

$$E = 2 \operatorname{Re} \left\{ \sum_{v=1}^2 T_v M_v B_v \right\}. \quad (1.2)$$

the notations adopted in [3, 4] are employed here and hereafter. The constants B_v are determined in terms of the magnitudes of the stresses to infinity.

Setting the displacement of the stiffener as a solid equal to zero, we obtain

$$U_2(u) = \int_0^u \frac{1}{E_0 F_0} dv_1 \int_{v_1}^l q(v) dv \quad (0 < u < l), \quad (1.3)$$

where E_0 and F_0 are the Young's modulus and the cross-sectional area of the stiffener, respectively. It can be assumed that E_0 and F_0 are variable quantities, but since an accounting of these factors is of no major significance, let us restrict ourselves to the case $E_0 = \text{const}$ and $F_0 = \text{const}$.

Substituting expressions (1.2) and (1.3) in Eq. (1.1) and inserting the variables

$H = lh$; $u = l\xi$; $v = l\eta$; $q(u) = \frac{P}{l} \varphi(\xi)$, we obtain

$$\varphi(\xi) + A \int_0^1 k(\xi, \eta) \varphi(\eta) d\eta + B \int_0^{\xi} (\eta - \xi) \varphi(\eta) d\eta = D - L\xi;$$

$$k(\xi, \eta) = \ln |\eta - \xi| + \frac{2}{C} \operatorname{Re} \left(\sum_{v=1}^2 T_v \{ l_v \bar{A}_1 \ln [(h + \bar{M}_1 \eta) - s_v(h + M_v \xi)] + n_v \bar{A}_2 \ln [(h + \bar{M}_2 \eta) - t_v(h + M_v \xi)] \} \right);$$

$$A = 2CGb l \Delta^{-1}; \quad B = Gb l^2 (E_0 F_0 \Delta)^{-1};$$

$$C = \operatorname{Re} \left\{ \sum_{v=1}^2 T_v A_v \right\}; \quad L = B(P_1 - E_0 F_0 E) P^{-1}. \quad (1.4)$$

after certain transformations. Solution of Eq. (1.4) should conform to the condition of equilibrium of the entire stiffener, which will assume the form

$$\int_0^1 \varphi(\eta) d\eta = \frac{P_1 - P_2}{P}. \quad (1.5)$$

with allowance for the substitution of variables. Equation (1.4) contains the unknown constant D , which can be determined from condition (1.5).

Let us make note of certain special cases.

When $h = \infty$ in Eq. (1.4),

$$k(\xi, \eta) = \ln |\xi - \eta|, \quad (1.6)$$

and we obtain the solution of the problem of the interaction between a stiffener and infinite anisotropic plate. The constants B_v are determined in terms of the quantities σ_x^∞ , σ_y^∞ , τ_{xy}^∞ . Having applied the limit conversion to the case of an isotropic medium, we find

$$A = \frac{(1+\nu)(\nu-3)Gbl}{4\pi\Delta h_1 E_1}$$

and we obtain, among other things, the solving equation of [5]. When $h = 0$ and $\theta = \pi/2$, the stiffener is located along the edge of the plate. In this case, the kernel $k(\xi, \eta)$ assumes the form of (1.6); in this case,

$$C = \operatorname{Re} \left\{ \sum_{\nu=1}^2 q_{\nu} (A_{\nu} + l_{\nu} \bar{A}_1 + n_{\nu} \bar{A}_2) \right\}.$$

If the plate material is significantly more rigid than that of the bonding layer and stiffener, we can set $A = 0$ in (1.4). The problem reduces to solution of the Volterra equation in the form

$$\varphi(\xi) = B \int_0^{\xi} (\xi - \eta) \varphi(\eta) d\eta + D - L\xi. \quad (1.7)$$

When $P_1 = P$, $P_2 = \sigma_x^{\infty} = \sigma_y^{\infty} = \tau_{xy}^{\infty} = 0$, the solution of (1.5) and (1.7) assumes the form:

$$\varphi(\xi) = \frac{\sqrt{B}}{\operatorname{sh} \sqrt{B}} \operatorname{ch} \sqrt{B}(1 - \xi).$$

If the stiffness of the bonding layer increases ($G/\Delta \rightarrow \infty$), we obtain the fundamental solving equation [3] and (1.4) by the limit conversion, which yields the solution of the problem under consideration for the case of a continuous joint between the stiffener and plate.

Let us note in conclusion that the solution of Eqs. (1.4) and (1.5) can be used to estimate the error of the results of strain measurement in low-modulus materials.

2. Expression (1.4) is an integral Fredholm equation of the second kind; $k(\xi, \eta)$ assumes logarithmic singularity when $\xi = \eta$. For $h = 0$ ($\theta \neq 0$), a stationary logarithmic singularity is also manifested when $\xi = \eta = 0$. The singularity of the kernel brings about the singularity of the solution $\varphi(\xi)$ of the integral equation (1.4). It can be shown [5, 6] that the asymptotic equations

$$\varphi'(\xi) \sim C_1 \ln \xi; \quad \xi \rightarrow +0; \quad \varphi'(\xi) \sim C_2 \ln(1 - \xi); \quad \xi \rightarrow 1-0,$$

are valid for $\varphi(\xi)$, where the constants C_1 and C_2 are defined in terms of the quantities $\varphi(0)$ and $\varphi(1)$ and the elastic parameters. In the case of a continuous joint between the stiffener and plate, $\varphi(\xi)$ assumes stepwise singularities at the ends [3]. The presence of a bonding layer leads to a reduction in the concentration of contact forces in the vicinity of the ends.

Equation (1.4) can be solved approximately by the method of a first-order collocation spline [7]. Let us present the desired solution in the form

$$\varphi_n(\xi) = \sum_{j=0}^n \varphi_j^n L_n^j(\xi), \quad (2.1)$$

where $L_n^j(\xi)$ are functions continuous in the interval $[0, 1]$ and linear in each interval $[(i-1)h, ih]$ ($i = \overline{1, n}$); $h = 1/n$; $L_n^j(ih) = \delta_{ji}$; and, δ_{ji} is a Kronecker delta. The unknown coefficients φ_j^n can be determined by the collocation method from the condition:

$$\varphi_n(ih) + A \int_0^1 k(ih, \eta) \varphi_n(\eta) d\eta + B \int_0^{ih} (\eta - ih) \varphi_n(\eta) d\eta = D - Lih$$

$$(i = \overline{0, n}),$$

or, what amounts to the same thing, from the system of linear algebraic equations:

$$\varphi_i^n + A \sum_{j=0}^n \left\{ \int_0^1 k(ih, \eta) L_n^j(\eta) d\eta \right\} \varphi_j^n + B \sum_{j=0}^i \left\{ \int_0^{ih} (\eta - ih) L_n^j(\eta) d\eta \right\} \varphi_j^n = D - Lih. \quad (2.2)$$

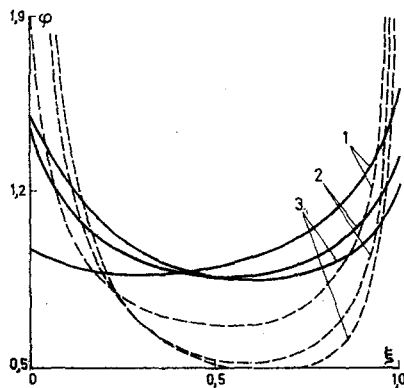


Fig. 1

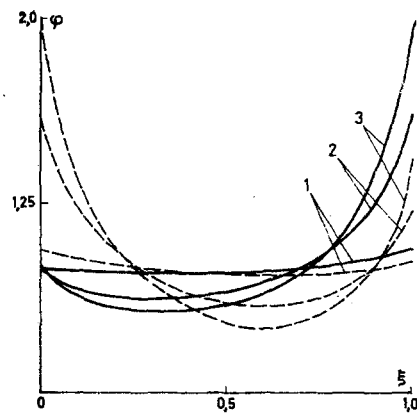


Fig. 2

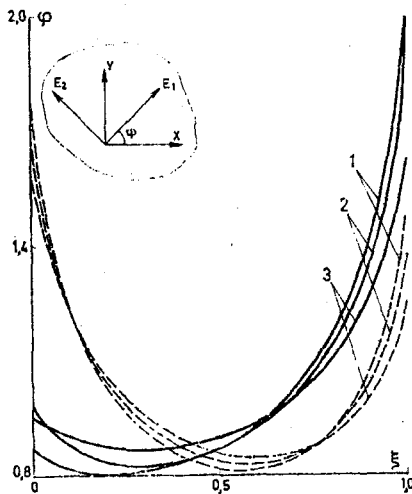


Fig. 3

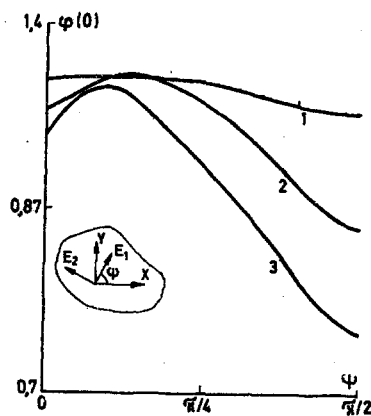


Fig. 4

The coefficients in the last system are computed in closed form. Still another equation, which is derived from supplementary condition (1.5), should attach to system (2.2).

3. The results of computation of $\varphi(\xi)$ for a plate fashioned from a glass-fiber/epoxide composite ($E_1 = 5.384 \cdot 10^4$ MPa, $E_2 = 1.795 \cdot 10^4$ MPa, $G_{12} = 0.863 \cdot 10^4$ MPa, and $\nu_{12} = 0.25$) with $P = P_1$ and $P_2 = 0$ and in the absence of tension to infinity are presented below. As independent parameters, we adopted $u = E_1 l h_1 / E_0 F_0$ (the relative stiffness of the rib), $v = G b l / E_1 h_1 \Delta$ (the relative stiffness of the bonding layer), h (the distance from the stiffener to the edge of the plate), and ψ (the angle formed between the principal direction of anisotropy of E_1 and the x axis).

The results of the computations indicated good convergence of the algorithm. The results of computations of $\varphi(0)$ and $\varphi(1)$ in the case $h = \psi = 0$ and $u = 0.5$ for different values of v are presented in Table 1. The interval $[0, 1]$ was divided into $N = 10, 20, 30$, and 40 equally spaced nodes.

The results of computations for the case of limit conversion to an isotropic material and a finite plate are in good agreement with data derived by Lubkin and Lewis [5]. Computed values of $\varphi(0)$ and $\varphi(1)$ for $N = 40$ and corresponding values from [5], which were obtained for $N = 150$, are presented in Table 2 for comparison. Comparison of the results indicates that the algorithm defined by Eqs. (2.1) and (2.2) is effective over a broad range of u and v .

The results of computations of $\varphi(\xi)$ for $h = 0, 1$, and ∞ (curves 1, 2, and 3, respectively) are presented in Fig. 1. Here, $\psi = 0$, $u = 0.5$, and $v = 2.0$; the solid lines correspond to a glued joint, and the broken lines to a continuous joint between the stiffener and plate [3]. The increase in h leads to an increase in the concentration of contact forces in the vicinity of the loaded end of the stiffener. As compared with the continuous, the glued joint produces a smoother distribution of contact forces over the length of the stiffener and significantly reduces the concentration of contact forces in the vicinity of the ends.

TABLE 1

ν	$N=10$		$N=20$		$N=30$		$N=40$	
	$\varphi(0)$	$\varphi(1)$	$\varphi(0)$	$\varphi(1)$	$\varphi(0)$	$\varphi(1)$	$\varphi(0)$	$\varphi(1)$
0,2	0,988	1,076	0,988	1,077	0,989	1,077	0,989	1,077
2,0	0,972	1,598	0,975	1,606	0,976	0,608	0,976	1,609
20,0	1,150	3,559	1,191	3,839	1,203	3,933	1,208	3,977

TABLE 2

B	A	$N=40$		$N=150$ [5]	
		$\varphi(0)$	$\varphi(1)$	$\varphi(0)$	$\varphi(1)$
1	2B	1,934	1,658	1,940	1,662
$10^{1/2}$	B/2	2,233	1,356	2,237	1,356
10	3B/16	3,338	1,098	3,349	1,095
$10^{2/3}$	B/16	5,633	0,629	5,663	0,628
10^2	B/64	9,994	0,219	10,010	0,219
$10^{5/2}$	9B/1600	17,549	0,118	17,790	0,125

Values of $\varphi(\xi)$ for different values of $\nu = 0, 2$, and 4 (curves 1, 2, and 3, respectively) are shown in Fig. 2; $\psi = 0$, and $u = 0.5$. Here and hereafter, the solid lines correspond to the case $h = 0$, and the broken lines to $h = \infty$. An increase in the thickness of the bonding layer or a reduction in its shear modulus gives rise to a decrease in the concentration of contact forces in the vicinity of the ends of the stiffener. Note that an increase in $E_0 F_0$ leads to an increase in the concentration of contact forces in the vicinity of the loaded end of the stiffener.

The results of computations of $\varphi(\xi)$ for different values of the angle $\varphi = 0, \pi/4$, and $\pi/2$ (curves 1, 2, and 3, respectively) are shown in Fig. 3 for $u = 0.5$ and $\nu = 2.0$. The dependence of $\varphi(0)$ on the angle ψ for the case $h = 0$, $u = 0.5$, and $\nu = 2.0$ is presented in Fig. 4. Curve 1 corresponds to the glass-fiber/epoxide composite, 2 to a boron-fiber/epoxide composite ($E_1 = 27.610 \cdot 10^4$ MPa, $E_2 = 2.761 \cdot 10^4$ MPa, $G_{12} = 1.035 \cdot 10^4$ MPa, and $\nu_{12} = 0.25$), and 3 to a graphite-fiber/epoxide composite ($E_1 = 27.610 \cdot 10^4$ MPa, $E_2 = 1.104 \cdot 10^4$ MPa, $G_{12} = 0.552 \cdot 10^4$ MPa, and $\nu_{12} = 0.25$). It is apparent that $\varphi(0)$ attains maximum values at certain values of the angle ψ in the interval $[0, \pi/2]$.

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