

The intensity factor K_I is always more (less) for that tip of the crack which is closer to the line of separation with the less (more) rigid medium (Fig. 2a, c). With a significant distance of the incision from the line of separation with the less rigid medium K_I has a maximum at $\alpha = 0$ and with approach of the separation to the boundary it is displaced to an angle of $\alpha = \pi/2$. If the more rigid medium is beyond the line of separation, then K_I is always a maximum at $\alpha = \pi/2$. In both cases the intensity factor K_{II} is greater for the more distant tip. It becomes a maximum with angles α close to $\pi/6$ (Fig. 2b, d).

The results obtained are similar to the known ones for a half-plane with a crack the edges of which are subject to a normal pressure when $G^+ \ll G^-$ [3].

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ANISOTROPY OF A PLATE WEAKENED BY AN ELLIPTICAL HOLE AND CRACKS

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At present significant successes have been attained in investigation of crack propagation close to holes in isotropic plates ([1-3] and the reviews contained in them). Problems for anisotropic plates have been little discussed [4, 5]. In this work special representations of the solution are constructed, and solving singular integral equations of the problem of the theory of elasticity for an anisotropic plate weakened by an elliptical hole and a system of incisions of arbitrary form are obtained. Automatic fulfillment of the boundary conditions on the contour of the hole provides an effective numerical solution of the problem in the case of internal and boundary incisions.

Let us consider an elastic anisotropic plate with an elliptical hole Λ and a system of k simple nonintersecting curved incisions L_j . Let us take the axes of symmetry of the ellipse as the coordinate axes of the complex variable $z = x + iy$. The plate is loaded with external forces $X_n + iY_n$ on the contour of the hole and σ_{ij}^0 at infinity. On the edges of the incision is specified the self-balanced load $X_n^\pm(t) + iY_n^\pm(t)$, $t \in L_j$ continuous according to Hölder. The sign "+" ("−") refers to the left (right) edge of the incision L_j in movement from its start a_j to end b_j (Fig. 1). All of the incisions are internal and the generalization for the case of incisions originating from the contour of the hole will be given below.

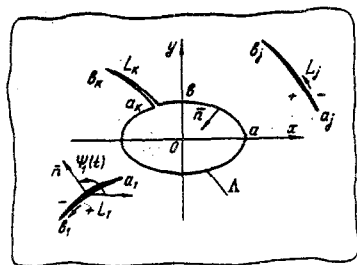


Fig. 1. Calculation plan.

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The stresses in the plate may be expressed through the two analytical functions $\Phi_\nu(z_\nu)$ ($\nu = 1, 2$):

$$(\sigma_x, \tau_{xy}, \sigma_y) = 2 \operatorname{Re} \left\{ \sum_{\nu=1}^2 (\mu_\nu^2, -\mu_\nu, 1) \Phi_\nu(z_\nu) \right\}, \quad (1)$$

where $z_\nu = x + \mu_\nu y$, μ_ν are the roots of the corresponding characteristic equation with the positive imaginary portions [6].

Let us represent the functions $\Phi_\nu(z_\nu)$ in the form

$$\Phi_\nu(z_\nu) = \sum_{j=1}^2 \Phi_{\nu j}(z_\nu). \quad (2)$$

Here $\Phi_{\nu 1}(z_\nu)$ determine the known basic stressed state caused by the external load on the contour of the hole and at infinity in a plate without incisions and $\Phi_{\nu 2}(z_\nu)$ the perturbed state occurring as the result of the presence of incisions.

Based on the solution of the problem of the action of a concentrated force at the point τ of an infinite anisotropic plate with an elliptical hole free of external forces with the semiaxes a and b [6, 7]

$$\begin{aligned} \Phi_\nu^*(z_\nu) &= \frac{1}{\omega'_\nu(\zeta_\nu)} \left\{ \frac{A_\nu}{\zeta_\nu - \eta_\nu} + \frac{l_\nu \bar{A}_1}{\zeta_\nu (\zeta_\nu \bar{\eta}_1 - 1)} + \frac{n_\nu \bar{A}_2}{\zeta_\nu (\zeta_\nu \bar{\eta}_2 - 1)} \right\}; \\ z_\nu = \omega_\nu(\zeta_\nu) &= \frac{a - i\mu_\nu b}{2} \zeta_\nu + \frac{a + i\mu_\nu b}{2} \zeta_\nu^{-1}; \quad |\zeta_\nu| > 1; \\ \zeta_\nu = \zeta_\nu(z_\nu) &= \frac{z_\nu + \sqrt{z_\nu^2 - (a^2 + \mu_\nu^2 b^2)}}{a - i\mu_\nu b}; \quad \eta_\nu = \zeta_\nu(\tau); \quad \zeta_\nu(\infty) = \infty; \\ l_\nu &= \frac{\mu_{3-\nu} + \bar{\mu}_1}{\mu_\nu - \mu_{3-\nu}}; \quad n_\nu = \frac{\mu_{3-\nu} - \bar{\mu}_2}{\mu_\nu - \mu_{3-\nu}}; \quad (\nu = 1, 2) \end{aligned}$$

and using the principle of superposition we will search for

$$\begin{aligned} \Phi_{\nu 2}(z_\nu) \text{ in the form } \left(L = \bigcup_{j=1}^k L_j \right): \\ \Phi_{\nu 2}(z_\nu) = \frac{1}{\omega'_\nu(\zeta_\nu)} \int_L \left\{ \frac{A_\nu(\tau)}{\zeta_\nu - \eta_\nu} + \frac{l_\nu \bar{A}_1(\tau)}{\zeta_\nu (\zeta_\nu \bar{\eta}_1 - 1)} + \right. \\ \left. + \frac{n_\nu \bar{A}_2(\tau)}{\zeta_\nu (\zeta_\nu \bar{\eta}_2 - 1)} \right\} ds = \frac{1}{2\pi i \omega'_\nu(\zeta_\nu)} \int_L \left\{ \frac{\Omega_\nu(\tau) d\tau_\nu}{\eta_\nu - \zeta_\nu} - \frac{l_\nu \bar{\Omega}_1(\tau) d\bar{\tau}_1}{\zeta_\nu (1 - \zeta_\nu \bar{\eta}_1)} - \right. \\ \left. - \frac{\eta_\nu \bar{\Omega}_2(\tau) d\bar{\tau}_2}{\zeta_\nu (1 - \zeta_\nu \bar{\eta}_2)} \right\}; \quad \Omega_\nu(t) = -2\pi i A_\nu(t)/M_\nu(t); \quad M_\nu(t) = \mu_\nu \cos \psi - \sin \psi, \end{aligned} \quad (3)$$

where $\Omega_\nu(t) = \{\Omega_{\nu j}(t) | t \in L_j, j = 1, k\}$ are unknown complex functions at the incision L , $\psi = \psi(t)$ is the angle between the normal $\bar{n}$ to the left edge of the crack L at the point t and the x axis, and $d\tau_\nu = M_\nu(t) ds$ (ds is an element of the arc L).

The functions $\Phi_{\nu 2}(z_\nu)$ constructed in such a manner automatically satisfy the conditions $X_n = Y_n = 0$ at the hole Λ and fade at infinity. Consequently selection of the function $\Phi_{\nu 2}(z_\nu)$ in the form of (2) and (3) provides fulfillment of the boundary conditions on the contour of the hole Λ and at infinity.

Substituting the limiting values of the function $\Phi_\nu(z_\nu)$ from Eqs. (2) and (3) in the boundary conditions at the incision L [4, 8]

$$\begin{aligned} a(t) \Phi_1^\pm(t_1) + b(t) \Phi_1^\pm(t_1) + \Phi_2^\pm(t_2) &= F^\pm(t); \\ a(t) &= \frac{\mu_1 - \bar{\mu}_2}{\mu_2 - \bar{\mu}_2} \cdot \frac{M_1(t)}{M_2(t)}; \quad b(t) = \frac{\mu_1 - \bar{\mu}_2}{\mu_2 - \bar{\mu}_2} \cdot \frac{\bar{M}_1(t)}{M_2(t)}; \end{aligned}$$

$$F^{\pm}(t) = \pm \frac{X_n^{\pm}(t) + i Y_n(t)}{(\mu_2 - \bar{\mu}_2) M_2(t)},$$

after certain transformations we obtain

$$\begin{aligned} \int_{L_j} \{K_1(t, \tau) \mathcal{Q}_1(\tau) + K_2(t, \tau) \overline{\mathcal{Q}_1(\tau)}\} ds &= f(t); \\ K_1(t, \tau) ds &= \frac{a(t)}{\pi i \omega'_1(\zeta_1)} \left\{ \frac{d\tau_1}{\eta_1 - \zeta_1} + \frac{n_1 \bar{b}(\tau) d\bar{\tau}_2}{\zeta_2(1 - \zeta_1 \bar{\eta}_2)} \right\} - \\ &- \frac{b(t)}{\pi i \omega'_1(\zeta_1)} \left\{ \frac{n_1 \bar{b}(\tau) d\tau_2}{\bar{\zeta}_1(1 - \bar{\zeta}_1 \eta_2)} - \frac{\bar{l}_1 d\tau_1}{\bar{\zeta}_1(1 - \bar{\zeta}_1 \eta_1)} \right\} - \\ &- \frac{1}{\pi i \omega'_2(\zeta_2)} \left\{ \frac{a(\tau) d\tau_2}{\eta_2 - \zeta_2} - \frac{n_2 \bar{b}(\tau) d\bar{\tau}_2}{\zeta_2(1 - \zeta_2 \bar{\eta}_2)} \right\}; \\ K_2(t, \tau) ds &= \frac{a(t)}{\pi i \omega'_1(\zeta_1)} \left\{ \frac{n_1 \bar{a}(\tau) d\bar{\tau}_2}{\zeta_1(1 - \zeta_1 \bar{\eta}_2)} - \frac{l_1 d\bar{\tau}_1}{\zeta_1(1 - \zeta_1 \eta_1)} \right\} - \\ &- \frac{b(\tau)}{\pi i \omega'_1(\zeta_1)} \left\{ \frac{d\tau_1}{\eta_1 - \zeta_1} + \frac{n_1 \bar{a}(\tau) d\tau_2}{\bar{\zeta}_1(1 - \bar{\zeta}_1 \eta_2)} \right\} - \\ &- \frac{1}{\pi i \omega'_2(\zeta_2)} \left\{ \frac{b(\tau) d\tau_2}{\eta_2 - \zeta_2} + \frac{l_2 d\bar{\tau}_1}{\zeta_2(1 - \zeta_2 \bar{\eta}_1)} - \frac{n_2 \bar{a}(\tau) d\bar{\tau}_2}{\zeta_2(1 - \zeta_2 \bar{\eta}_2)} \right\}; \\ f(t) &= F_2(t) - 2 \{a(t) \Phi_{11}(t_1) + b(t) \overline{\Phi_{11}(t_1)} + \\ &+ \Phi_{21}(t_2)\} + \frac{a(t) n_1}{\pi i \omega'_1(\zeta_1)} \int_{L_j} \frac{\bar{F}_1(\tau) d\bar{\tau}_2}{\zeta_1(1 - \zeta_1 \bar{\eta}_2)} - \frac{b(t) \bar{n}_1}{\pi i \omega'_1(\zeta_1)} \int_{L_j} \frac{\bar{F}_1(\tau) d\tau_2}{\bar{\zeta}_1(1 - \bar{\zeta}_1 \eta_2)} - \\ &- \frac{1}{\pi i \omega'_2(\zeta_2)} \int_{L_j} \left\{ \frac{F_1(\tau) d\tau_2}{\eta_2 - \zeta_2} - \frac{n_2 \bar{F}_1(\tau) d\bar{\tau}_2}{\zeta_2(1 - \zeta_2 \bar{\eta}_2)} \right\}; \\ \mathcal{Q}_2(t) &= F_1(t) - a(t) \mathcal{Q}_1(t) - b(t) \overline{\mathcal{Q}_1(t)}; \quad F_{1,2}(t) = F^+(t) \mp F^-(t). \end{aligned} \tag{4}$$

In addition to Eqs. (4) conditions of uniqueness of the displacements along each crack L_j must be fulfilled:

$$\begin{aligned} \int_{L_j} \mathcal{Q}_1(\tau) d\tau_1 &= A \left\{ \operatorname{Re} \left[q_2 \int_{L_j} F_1 d\tau_2 \right] - \mu_1 \operatorname{Re} \left[p_2 \int_{L_j} F_1 d\tau_2 \right] \right\}; \\ A &= 2 \{a_{11}(\mu_1 - \mu_2)(\mu_1 - \bar{\mu}_2)(\mu_1 - \bar{\mu}_1)\}^{-1}. \end{aligned} \tag{5}$$

Here

$$p_v = a_{11} \mu_v^2 - a_{16} \mu_v + a_{12}; \quad q_v = a_{12} \mu_v + a_{22} \mu_v^{-1} - a_{26};$$

and a_{ij} are the coefficients of the deformations from Hooke's law.

According to the assumptions relative to the smoothness of the incisions L_j the solution of system (4), the index of which is equal to (+1), with the additional limitations of (5) in the class of functions

$$\mathcal{Q}_1(t) = \chi^j(\xi) (1 - \xi^2)^{-1/2}, \quad t \in L_j = \{t = \tau^j(\xi) \mid |\xi| < 1\}, \tag{6}$$

where $\chi^j(\xi)$ are limited continuous functions according to Hölder in the section $[-1, +1]$, exists uniquely [9]. With the use of the Gauss-Chebyshev quadratic equations the system of singular integral Eqs. (4) with the additional conditions (5) reduces to an algebraic system of equations of the relatively approximate values of the sought-for functions $\chi^j(\xi)$ at the Chebyshev nodes [1]. Having determined $\chi^j(+1)$ by asymptotic formulas [9], we find the distribution of stresses and the opening and edge-sliding mode stress intensity factors

$$K_I(c) = \lim_{t \rightarrow c} \sigma_n \sqrt{2\pi r}; \quad K_{II}(c) = \lim_{t \rightarrow c} \tau_n \sqrt{2\pi r}$$

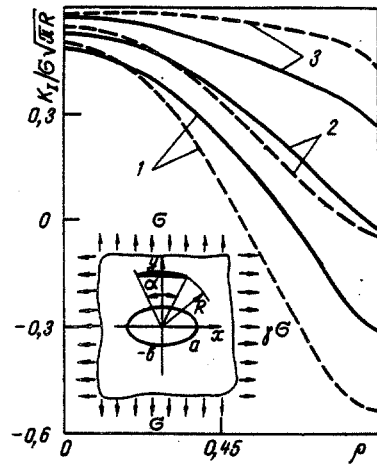


Fig. 2

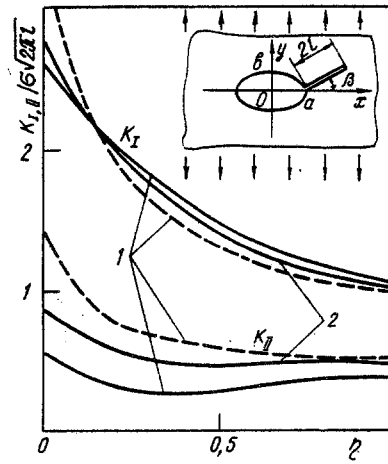


Fig. 3

Fig. 2. Relationship of the rupture stress intensity factor K_I for a crack located along an arc of the circumference in a plate of material 1 to the parameter $\rho = b/R$ with $\alpha = \pi/4$, $\lambda = 1$, and $\gamma = -1, 0$, and 1 (curves 1, 2, and 3 respectively), $\phi = 0$ (solid lines), and $\phi = \pi/2$ (broken).

Fig. 3. Relationship of the opening mode K_I (upper curves) and edge-sliding mode K_{II} (lower curves) stress intensity factors at the tip of an edge crack in a plate of materials 1 and 2 (curves 1 and 2, respectively) to the parameter $\eta = (2l \cos \beta)/a$ with $\beta = \pi/6$, $\lambda = 1$, and $\phi = 0$ (solid lines) and $\phi = \pi/2$ (broken).

at the tip of the crack L_j . Here t is a point lying on the extension of the crack beyond its tip c on the tangent and $r = |t - c|$.

Below are presented the results of calculations for plates of orthotropic materials with different values of $\lambda = b/a$ and of the angle ϕ formed by the main direction of anisotropy E_1 with the x -axis. The parameters of anisotropy of the materials with $\phi = 0$ were: 1) $E_1/E_2 = 3$, $\mu_1 = 2.27i$, $\mu_2 = 0.76i$; 2) $E_1/E_2 = 1$, $\mu_1 = 1.04i$, $\mu_2 = 0.96i$ (an almost isotropic material).

The angle of anisotropy ϕ has a significant influence on the stress intensity factor K_I with $\rho > 0.3$ (Fig. 2). The negative values of K_I in the presence of a compressive load along the x -axis (curves 1) indicate partial contact of the crack edges. According to the data of calculations for a plate of material 1 with $\gamma = 1$ and $\rho = 0$ with $N = 30$ $K_I = 0.584\sqrt{\pi R}$ corresponds to the Chebyshev nodes of division of the incision $L = \{t = R \exp[i\pi(4-\xi)/8] \mid |\xi| < 1\}$, which coincides at all significant digits with the accurate solution of the problem of a crack L along an arc of a circumference in an isotropic plate without a hole [2].

In the case of entry of a crack onto the contour of a hole conditions (5) providing uniqueness of the displacements in by-passing of the corresponding contours are not fulfilled and are rejected. The potentials (2) and Eqs. (4) must be determined more accurately by the known method [2]. Below a simplified procedure of numerical solution based on representation of the solution in form (6) is used for edge cracks [2]. If the crack L_j enters onto the contour of the hole at the end $\tau_j(+1)$, then we take $\chi_j(+1) = 0$ as an additional condition. This simplified method is ineffective if it is necessary to determine the stress distribution close to the point of entry of a crack onto the contour of a hole but provides completely satisfactory results for the inner tip of the crack.

The factors K_I and K_{II} increase with a decrease in the length of an edge crack (Fig. 3). The spread between the values of K_I with $\phi = 0$ and $\phi = \pi/2$ is especially significant with $\eta < 0.1$ and increases with an increase in the degree of anisotropy of the material E_1/E_2 . The data for an isotropic material occupied an intermediate position between the corresponding values of K_I with $\phi = 0$ and $\phi = \pi/2$. A comparison was made of the numerical results obtained (Fig. 3) with $N = 10$ and 20 to the Chebyshev nodes of division of the contour of an edge crack with data for an isotropic plate with $\lambda = 0.5$ and $\phi = 0$ [10] (Table 1).

TABLE 1

β	η	$K_{II}/\sigma \sqrt{\pi \eta}$			$K_{II}/\sigma \sqrt{\pi \eta}$		
		$N=10$	$N=20$	according to data of [10]	$N=10$	$N=20$	according to data of [10]
$\pi/12$	0,01	5,033	5,038	5,038	0,781	0,780	0,779
	0,1	3,172	3,174	3,175	0,429	0,428	0,428
	1,0	1,182	1,182	1,182	0,297	0,297	0,296
$\pi/4$	0,01	4,051	4,027	4,017	1,936	1,938	1,938
	0,1	2,571	2,562	2,559	1,076	1,074	1,074
	1,0	0,709	0,709	0,709	0,747	0,747	0,747

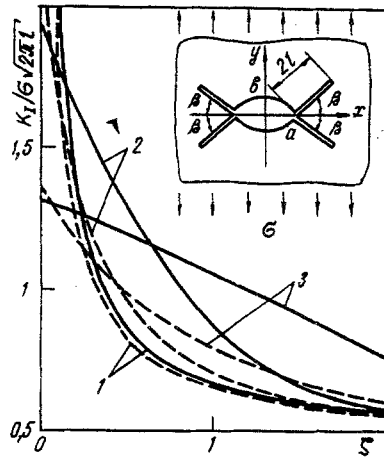


Fig. 4

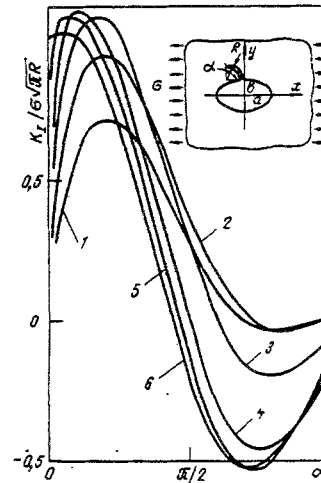


Fig. 5

Fig. 4. Relationship of the opening mode stress intensity factor K_I at the tips of four cracks in a plate of material 1 to the relative crack length $\zeta = 2l/a$ with $\beta = \pi/4$, $\lambda = 0, 1$, and 2 (curves 1-3, respectively), and $\phi = 0$ (solid lines) and $\phi = \pi/2$ (broken).

Fig. 5. Influence of the angle of opening α of an edge crack located along the arc of a circumference $L = \{t = R[\exp(i\alpha \times (\xi + 1)/2) - 1] + ib \mid |\xi| < 1\}$ in a plate of material 2 on the value of K_I with $R/b = 1$ and $\lambda = 0, 0.2, 0.5, 1, 2$, and ∞ (curves 1-6, respectively).

Even with $N > 10$ the results of the calculations do not change in the first two significant digits, which is an indication of the good convergence of the algorithm. The spread does not exceed 1%.

In the case of four symmetric edge cracks (Fig. 4) in compression of an elliptical hole along the y axis (a decrease in λ) the values of the stress intensity factor K_I at the tips of small ($\zeta < 0.1$) cracks increase sharply with a decrease in their length. With $\zeta > 1$ the form of the hole has a weak influence on the stress intensity factor K_i ($i = I, II$) at the crack tips. With $\lambda = 0$ the results of calculations of K_i for material 2 agree with the data of calculation of an isotropic plate with a five-link broken crack [2].

The maximum value of K_I at the tip of a curved edge crack occurs with angles of crack opening α close to $3\pi/20$ and the minimum with $\alpha \sim 3\pi/4$ (Fig. 5). For angles of $\alpha > \pi/2$ the factor K_I is negative as the result of contact of the crack edges. With $\lambda = 0$ (curve 1) the value of K_I coincides with that obtained in solution of the problem of an edge crack located along the arc of circumference in an isotropic half-plane [11].

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INFLUENCE OF NONUNIFORM SURFACE TENSION ON THE DIFFUSION AND STRESSED STATE OF A SOLID

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UDC 539.3

The presence of an impurity [1], the redistribution of its concentration, and the stressed state created [2] cause nonuniform surface tension [3]. Let us investigate the local stresses occurring and the change in impurity concentration in a solid and in its thin surface layer.

Let us use a model [3] taking into consideration the difference in properties of the surface layer of a real body from the properties of the remaining material, in which the solid is simulated by an elastic half-space $z \geq 0$ covered with a two-dimensional continuum Σ with its own physicomaterial characteristics.

Let us assume that the nonuniformity of the surface tension is local, its dimensions are small in comparison with the dimensions of the body, and its surface tension σ_Σ changes in a jump as does the chemical potential of the dissolved substance ϕ_Σ in the surface layer

$$\sigma_\Sigma = \gamma_2 + (\gamma_1 - \gamma_2)S(r-R), \quad (1)$$

$$\phi_\Sigma = \phi_0 S(R-r), \quad (2)$$

where γ_2 and γ_1 are the surface tension in the area $r < R$ and outside it, respectively, $S(x)$ is a Heaviside function, r is the radial coordinate, and ϕ_0 is the chemical potential in the area of the radius R on the surface $z = 0$.

The original system for studying the diffusion of the impurity and the stressed state occurring in the solid with nonuniform surface tension is relationships [3] written for a quasistatic formulation of the problem in taking into consideration the interrelationship of

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