

STRESS STATE NEAR LOADED HOLES IN ANISOTROPIC PLATES

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Special representations of the solution are constructed and solving integral equations of the problem of the elastic equilibrium of a finite anisotropic plate weakened by an elliptical hole or rectilinear crack are derived. The absence of the unknown function for the boundary of the internal hole (crack) makes it possible to propose an effective algorithm for the problem's numeric solution. The results of calculations, which illustrate the effect of the external boundary and material anisotropy on the stress distribution near loaded holes of different sizes, are presented. Direct comparison with the finite-element method indicates that the proposed algorithm significantly lowers the amount of input data, the computer time, and the required volume of memory with comparable accuracy.

The method of integral equations (MIE) is widely used to analyze the stress state of finite multicoupled plates with cracks [1]. Problems for isotropic plates are rather well understood. In this case, the order of the system of developing integral equations is equal to the number of components in the body's boundary. In our study, we construct special representations of the solution and derive the solving equation of the problem of the elastic equilibrium of a finite anisotropic plate weakened by an elliptical hole or rectilinear crack. The absence of an unknown function on the boundary of the internal opening (crack) enables us to propose an effective algorithm for the problem's solution.

Let us examine the elastic equilibrium of an anisotropic plate bounded by the closed contours Λ (an ellipse with semi-axes a and b) and C (an arbitrary smooth external contour) due to a self-balanced system of external forces applied to $\Gamma = C \cup \Lambda$. Let us combine the axes of symmetry of the ellipse with the axes of the cartesian coordinate system xy . The direction leaving the plate from the left can be selected as the positive direction for Γ . The normal $\bar{n}$ can be directed to the right for a positive tracing of Γ .

The stresses in the plate are expressed in terms of two analytic functions $\Phi_\nu(z_\nu)$ ($z_\nu = x + \mu_\nu y$, $\nu = 1, 2$), which satisfy the edge conditions on Γ [2],

$$a(t)\Phi_1^+(t_1) + b(t)\overline{\Phi_1^+(t_1)} + \Phi_2^+(t_2) = F(t), \quad t \in \Gamma,$$

where

$$\begin{aligned} F(t) &= [X_n(t) + \bar{\mu}_2 Y_n(t)]/[\mu_2 - \bar{\mu}_2] M_2(t); \quad a(t) = a_0 M_1(t)/M_2(t); \\ b(t) &= b_0 \bar{M}_1(t)/M_2(t); \quad a_0 = (\mu_1 - \bar{\mu}_2)(\mu_2 - \bar{\mu}_2)^{-1}; \\ b(t) &= (\bar{\mu}_1 - \mu_2)(\mu_2 - \mu_2)^{-1}; \quad M_\nu(t) = \mu_\nu \cos \psi - \sin \psi; \end{aligned} \quad (1)$$

the superscript "+" denotes the limiting value of the function on approaching the contour Γ from the left, $X_n(t)$ and $Y_n(t)$ are the projections of the external forces at point $t \in \Gamma$ on the x and y axes, respectively, and $\psi = \psi(t)$ is the angle between the normal $\bar{n}$ at point $t \in \Gamma$ and the x axis.

Proceeding from the solution of the problem of a concentrated force and infinite anisotropic plate with an elliptical hole [3] and using the results obtained in [2, 4], the desired functions can be found in the form

TABLE 1

w/a	e/a	φ	M			
			10	20	30	40
3	2	0	1,851	1,859	1,863	1,865
			-0,691	-0,698	-0,699	0,699
		$\pi/2$	3,413	3,452	3,466	3,468
	1,5	0	-0,680	-0,693	-0,698	-0,701
			2,209	2,231	2,239	2,243
		$\pi/2$	-0,946	-0,954	-0,957	-0,958
2	2	0	4,250	4,147	4,203	4,233
			-1,029	-0,999	-1,018	-1,029
		$\pi/2$	2,596	2,516	2,518	2,514
	1,5	0	-0,676	-0,695	-0,694	-0,695
			4,191	4,229	4,2512	4,263
		$\pi/2$	-0,676	-0,695	-0,699	-0,701
		0	3,003	3,266	3,189	3,189
			-0,965	-1,027	-0,998	-0,993
		$\pi/2$	5,169	5,294	5,352	5,383
			-0,976	-1,009	-1,024	-1,032

$$\Phi_v(z_v) = \sum_{j=0}^1 \Phi_v^j(z_v); \quad \Phi_v^1(z_v) = [2\pi i \omega_v(\xi_v)]^{-1} \int_C \left\{ \frac{\Omega_v(\tau) d\tau_v}{\eta_v - \xi_v} + \frac{l_v \overline{\Omega_1(\tau)} d\bar{\tau}_1}{\xi_v(\xi_v \eta_1 - 1)} + \right. \\ \left. + \frac{n_v \overline{\Omega_2(\tau)} d\bar{\tau}_2}{\xi_v(\xi_v \eta_2 - 1)} \right\}; \quad \Omega_2(t) = -a(t) \Omega_1(t) + b(t) \overline{\Omega_1(t)}. \quad (2)$$

The notations adopted by Maksimenko [4] are used here and below, $\Omega_1(t)$ are unknown complex functions for C , and $\Phi_v^0(z_v)$ is the solution for an infinite anisotropic plate with an elliptical hole Λ due to the effect of external forces along its contour.

The functions $\Phi_v^0(z_v)$ can be represented using familiar methods in the form of the sum of an infinite series or Cauchy-type integrals, whose kernels and densities are solutions to the problem of a concentrated force applied to the perimeter of a hole, and the projections of external forces on the normal and tangent to the contour $\Lambda = \{z = a \cos \gamma + i b \sin \gamma \mid 0 \leq \gamma \leq 2\pi\}$ $p(\gamma)$, $q(\gamma)$. Thus, the direct application of the resultant representations $\Phi_v^0(z_v)$ for calculation of the stress-strain state (SSS) near the contour Λ of a hole is fraught with familiar computational difficulties associated with calculation of series coefficients and Cauchy-type integrals in the vicinity of the integration contour. To overcome these difficulties, $p(\gamma)$ and $q(\gamma)$ can be approximated by the piecewise-constant expressions

$$p(\gamma) \approx p_j; \quad q(\gamma) \approx q_j; \quad \gamma_{j-1} \leq \gamma \leq \gamma_j; \quad \gamma_j = j \Delta \gamma; \quad \Delta \gamma = 2\pi/n \quad (j = \overline{0, n}).$$

This can always be done with a sufficient degree of accuracy. The following closed analytic representations can then be obtained for $\Phi_v^0(z_v)$ after certain transformations:

$$\Phi_v^0(z_v) = [z_v'(\xi_v)]^{-1} \sum_{j=1}^n \{p_j \Phi_{v1}^j(z_v) + q_j \Phi_{v2}^j(z_v)\}; \quad \Phi_{v1}^j(z_v) = -\frac{A_{v2}^p}{\xi_v} + \\ + \frac{M_{vj}^p}{(\mu_v - \mu_{3-v})(\sigma_j - \xi_v)} - \frac{i}{4\pi(\mu_v - \mu_{3-v})} \left\{ [a_p \Delta_j + b_j \delta_j] s_{vj} + b_{vp} [\ln t_{vj} - \xi_v s_{vj}] + c_{vp} \left[\frac{s_{vj}}{\xi_v} + \frac{1}{\xi_v^2} (\ln t_{vj} - \ln m_j) \right] \right\}; \\ a_1 = a \mu_{3-v}; \quad a_2 = a; \quad b_1 = -bi; \quad b_2 = b \mu_{3-v} i; \quad \mu_v = \alpha_v + i \beta_v; \\ \Delta_j = \sigma_{j-1} + \sigma_{j-1}^{-1}; \quad \delta_j = \sigma_{j-1} - \sigma_{j-1}^{-1}; \quad s_{vj} = \frac{\sigma_j - \sigma_{j-1}}{(\sigma_j - \xi_v)(\sigma_{j-1} - \xi_v)}; \\ b_{v1} = a \mu_{3-v} - bi; \quad b_{v2} = b \mu_{3-v} i + a; \quad t_{vj} = (\sigma_{j-1} - \xi_v)/(\sigma_j - \xi_v); \\ c_{v1} = a \mu_{3-v} + bi; \quad c_{v2} = b \mu_{3-v} i - a; \quad m_j = \sigma_{j-1}/\sigma_j; \\ M_{vj}^p = 2i [\mu_{3-v} \operatorname{Im}(A_{vj}^p + A_{2j}^p) - \sum_{k=1}^2 (\beta_k \operatorname{Re} A_{kj}^p + \alpha_k \operatorname{Im} A_{kj}^p)]; \\ z_v = z_v(\xi_v) = \frac{a + i \mu_v b}{2} \xi_v + \frac{a - i \mu_v b}{2} \frac{1}{\xi_v}; \quad |\xi_v| < 1; \\ \xi_v = \xi_v(z_v) = \frac{z_v - \sqrt{z_v^2 - (a^2 + \mu_v^2 b^2)}}{a + i \mu_v b}; \quad \xi_v(\infty) = 0, \quad (3)$$

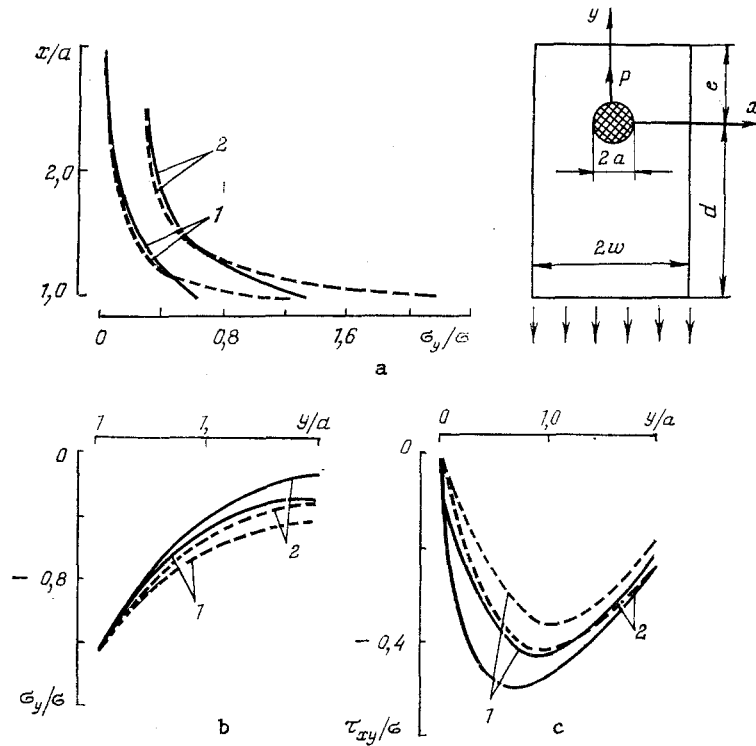


Fig. 1

where $\sigma_j = \exp(-i\gamma_j)$, and A_{vj}^p are determined from the system of equations

$$\sum_{v=1}^2 (\mu_v^{k-2} A_{vj}^p + \overline{\mu_v^{k-2}} \overline{A_{vj}^p}) = b_k^p / (2\pi i) \quad (k = \overline{1, 4});$$

$$b_1^p = \frac{-a_{12}b_3^p + a_{26}b_2^p}{a_{22}}; \quad b_2^1 = \frac{a}{2} (\Delta_j - \Delta_{j+1}); \quad b_2^2 = \frac{ib}{2} (\delta_j - \delta_{j+1});$$

$$b_3^1 = -b_2^2; \quad b_3^2 = -b_2^1; \quad b_4^p = \frac{a_{16}b_3^p + a_{12}b_2^p}{a_{11}}.$$

$\Phi_{vp}^j(z_v)$ are the solutions for a plate with a hole loaded along the portion $[z(\gamma_{j-1}), z(\gamma_j)]$ of the contour Λ by unit normal ($p = 1$) and tangential ($p = 2$) forces.

Thus, the plotted potentials $\Phi_v(z_v)$ of (2) and (3) automatically satisfy edge conditions (1) on Λ . Substituting the limiting values of the functions $\Phi_v(z_v)$ from (2) in edge conditions (1) on C , we obtain the following integral Fredholm equation of the second kind for the unknown function $\Omega_1(t)$ on C after certain transformations:

$$2b(t)\Omega_1(t) + \int_C \{K_1(t, \tau)\Omega_1(\tau) + K_2(t, \tau)\overline{\Omega_1(\tau)}\} ds = f(t);$$

$$K_1(t, \tau) ds = \frac{a(t)}{\pi i \omega_1'(\xi_1)} \left\{ \frac{d\tau_1}{\eta_1 - \xi_1} - \frac{n_1 \overline{b(\tau)} d\tau_2}{\xi_1(1 - \xi_1 \eta_2)} \right\} + \frac{b(t)}{\pi i \omega_1'(\xi_1)} \left\{ \frac{\overline{l_1} d\tau_1}{\xi_1(1 - \xi_1 \eta_1)} - \frac{\overline{n_1} a(\tau) d\tau_2}{\xi_1(1 - \xi_1 \eta_2)} \right\} - \frac{1}{\pi i \omega_2'(\xi_2)} \left\{ \frac{a(\tau) d\tau_2}{\eta_2 - \xi_2} + \frac{n_2 \overline{b(\tau)} d\tau_2}{\xi_2(1 - \xi_2 \eta_2)} \right\};$$

$$K_2(t, \tau) ds = \frac{a(t)}{\pi i \omega_1'(\xi_1)} \left\{ \frac{l_1 d\tau_1}{\xi_1(\xi_1 \eta_1 - 1)} + \frac{n_1 \overline{a(\tau)} d\tau_2}{\xi_1(1 - \xi_1 \eta_2)} \right\} - \frac{b(t)}{\pi i \omega_1'(\xi_1)} \left\{ \frac{d\tau_1}{\eta_1 - \xi_1} - \frac{n_1 \overline{b(\tau)} d\tau_2}{\xi_1(1 - \xi_1 \eta_2)} \right\} - \frac{1}{\pi i \omega_2'(\xi_2)} \left\{ \frac{l_2 d\tau_1}{\xi_2(1 - \xi_2 \eta_1)} - \frac{b(\tau) d\tau_2}{\eta_2 - \xi_2} - \frac{n_2 \overline{a(\tau)} d\tau_2}{\xi_2(1 - \xi_2 \eta_2)} \right\};$$

$$f(t) = 2 \{F(t) - a(t)\Phi_1^0(t_1) - b(t)\Phi_1^0(t_1) - \Phi_2^0(t_2)\};$$

$$d\tau_v = M_v(\tau) ds.$$
(4)

As was done by Sherman [2], it can be shown that resultant integral equation (4) has a unique solution.

The limiting case $b = 0$ ($a = 0$) yields a solution to the problem for a finite point with a rectilinear cut $\Lambda = \{ |x| < a; y = 0 \}$ ($\Lambda = \{ x = 0; |y| < b \}$).

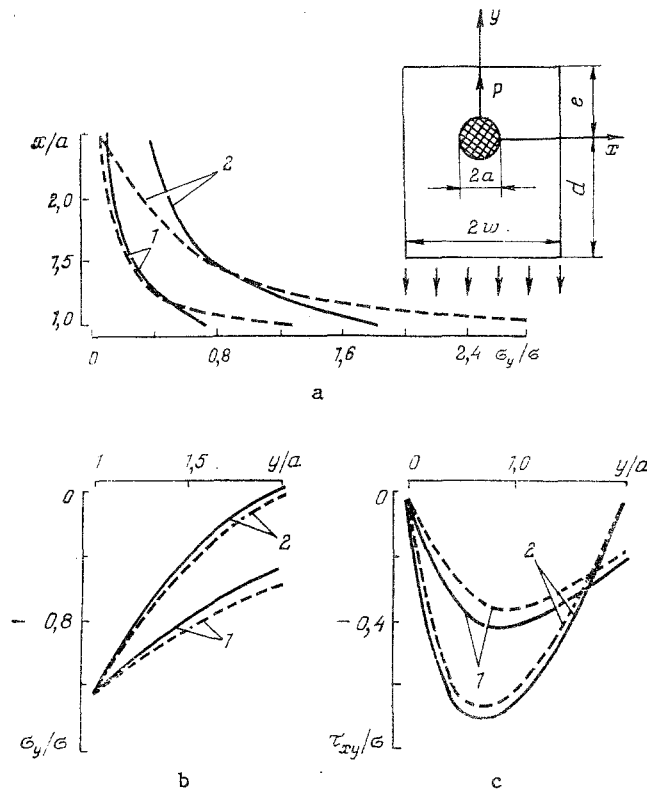


Fig. 2

Using the equations of rectangles for the integrals over C , the solution of Eq. (4) can be reduced to the solution of a system of linear algebraic equations in terms of approximate values of the functions $\Omega_1(t)$ at arbitrarily selected nodal points $t_k \in C$ ($k = 1, \dots, N$; N is the number of partitioning nodes on C). Taking the problem's symmetry with respect to one or two axes into account, the number of linear algebraic equations to which integral equation (4) is reduced, can be lowered by a factor of two or four, respectively.

The numeric values of stresses near free and loaded holes in rectangular anisotropic and isotropic plates, which were obtained by the proposed method, are given below. The data for the isotropic material were obtained by limiting conversion to parameters in the numeric solution.

Let us examine a rectangular plate with a thickness h , which occupies the region $D = \{ |x| < w; -d < y < e \}$ and which is weakened by a hole of radius a with a center at the origin of coordinates. The plate is loaded via a rigid pin, inserted into the hole by a force $P = 2ah\sigma$ (σ is the bearing stress) along the y axis (problem 1) or tensioned uniformly by distributed forces applied to the edges $y = e$ and $-d$ (problem 2). In the first case, balancing uniformly distributed forces $\sigma_1 = P/(2wh)$ are applied to the lower edge of the plate $y = -d$. As simplifying assumptions in problem 1, we will assume that friction is absent in the zone of contact between the pin and plate, and the transfer of force from the pin is modeled by a normal pressure distributed in accordance with the cosine law $p(t) = (4\sigma/\pi)\cos\beta$ (β is the angle between point t on the contour of the hole and the y axis).

The results of the computations of the stresses near a loaded hole in a fiberglass plate with the elastic parameters $E_1 = 53.84$ GPa, $E_2 = 17.96$ GPa, $G_{12} = 8.63$ GPa, and $\nu_1 = 0.25$ are presented in Figs. 1 and 2 and in Table 1. The solid (broken) lines apply to the case when the angle formed by the principal direction of anisotropy E_1 and the x axis is 0 ($\pi/2$).

Figure 1 illustrates the effect of bandwidth $2w$ on the distribution of stresses σ_y and τ_{xy} along the x and y axes (Fig. 1a and 1b) and the shear line $x = a$ (Fig. 1b) when $e/a = 6$, $d/a = 12$, and $w/a = \infty$ and 3 (curves 1 and 2). For plates of finite width ($w = 3a$), the stress concentration in the net section (Fig. 1a) increases by a factor of nearly two as compared with the case of an infinite plate ($w/a = \infty$). The maximum shear stresses (Fig. 1c) depend to a lesser degree on the width of the plate and are increased by no more than 20%. This agrees well with the fact that narrow plates with loaded holes fail, as a rule, through the net-section, and not due to bearing stress or shear.

The effect of the distance e to the free edge of the plate on stress distribution near a loaded hole is shown in Fig. 2 for $w/a = 3$, $d/a = 12$, and $e/a = 6.2$ (curves 1 and 2). Both the maximum tensile stresses in the net section (Fig. 2a) and

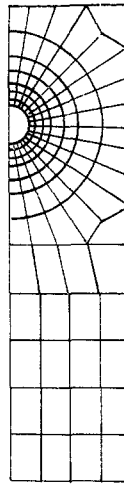


Fig. 3

the maximum shear stresses (Fig. 2c) increase significantly when $e/a = 2$ as compared with the case $e/a = 6$. For small e/a , the shear stresses become critical.

The results of calculations of $\sigma_y(a, 0)/\sigma$ (above the line) and $\tau_{xy}(a, a/2)/\sigma$ (below the line) for $d/a = 6$, $\varphi = 0$, and $\pi/2$, $w/a = 2$ and 3 , $e/a = 1.5$ and 2 , and different M values (the number of uniformly distributed nodes on the right half of C) are presented in Table 1. In the parameter ranges under consideration, M should be assigned a number of approximately 20 to obtain a solution with an accuracy to 1%. Convergence improves with increasing e/a and w/a .

To compare the effectiveness of the proposed algorithm with calculations performed in accordance with the finite-element method (FEM), let us turn to problem 2 for an isotropic plate with $e/a = w/a = 5$ and $d/a = 15$. The comparison is based on the overall partitioning of boundary C : within the framework of the proposed approach, half of the contour was partitioned into $M = 21$ boundary elements, corresponding to 134 quadratic finite elements (see Fig. 3). The unknown function was assumed constant within the limits of each boundary element. For this case, the stress concentration is known and is 3.14 [5]. Calculations based on the finite-element method and method of integral equations yield 3.11 and 3.15, respectively. As compared with the FEM, the MIE permits an approximate tenfold reduction in computation time on the ES-1046 computer.

Potentials (2) and (3) were found to be an effective instrument for determination of the SSS near loaded holes in finite anisotropic plates of complex shape. The described approach is distinguished by simplicity, and as compared with the finite-element method, makes it possible to reduce significantly the amount of input data, the time required for calculation on a computer, and the required volume of computer memory. Note that according to Maksimenko [4], the results obtained can be used to estimate crack growth near loaded holes in finite anisotropic plates.

REFERENCES

1. M. P. Savruk, Two-Dimensional Problems of the Theory of Elasticity for Cracked Bodies [in Russian], Kiev (1981).
2. D. I. Sherman, "Solution of the plane problem of the theory of elasticity for an anisotropic medium," *Prikl. Mat. Mekh.*, **6**, No. 6, 509-514 (1942).
3. A. S. Kosmodamianskii, Stress State of Anisotropic Media with Holes or Recesses [in Russian], Kiev, Donetsk (1976).
4. V. N. Maksimenko, "Limiting equilibrium of an anisotropic plate weakened by an elliptical hole and a system of cracks of complex shape," *Uch. Zap. Tsentr. Aërogidrodinam. Inst. (TsAGI)*, **18**, No. 3, 105-113 (1987).
5. R. Peterson, Stress Concentration Factors [Russian translation], Moscow (1977).