

ELASTIC EQUILIBRIUM OF A FINITE ANISOTROPIC PLATE WEAKENED BY AN ELLIPTICAL HOLE AND CONTAINING CRACKS

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The loss of load-carrying capacity by structural elements is often due to the presence of stress raisers such as holes and cracks in the elements. The publications [7, 9] offer a survey of the use of the method of integral equations to analyze stresses in finite plates weakened by holes and cracks of fairly general configuration. Problems for isotropic plates have been most frequently investigated by this approach. There are almost no solutions for anisotropic plates.

Below, we construct special representations of the solution and derive the resolvent system of integral equations for the problem of the elastic equilibrium of a finite linearly-anisotropic plate weakened by an elliptical hole and a system of curvilinear cracks. The absence of an unknown function at the boundary of the internal hole makes it possible to suggest the use of an efficient algorithm for numerical solution of the problem. As an example, we solve the problem for circular and rectangular plates with cracks reaching the contour of an internal hole. We study the effect of orthotropy of the material, the form of the hole, and the external boundary on the stress intensity factor (SIF) at the tips of the cracks.

1. Let the region D occupied by an elastic anisotropic plate be bounded by the closed contours Λ (an ellipse with semi-axes a and b) and L_0 (an arbitrary external contour) and contain n curvilinear nonintersecting slits L_j ($j = 1, n$) (Fig. 1). We make the symmetry axis of the ellipse coincide with the axes of the cartesian coordinate system xOy . As the positive direction on L_0, Λ , we take the direction to the left of region D . The direction on L_j is arbitrary. We direct the normal n to the right with positive circumvention of Λ, L_j ($j = 1, n$). We will assume that the contours L_j represent Liapunov curves. We

designate $L = \bigcup_{j=0}^n L_j, L' = L \setminus L_0$.

We will limit ourselves to the case when the load $X_n(t) + iY_n(t)$ applied to each closed contour is self-balanced. Meanwhile, all of the slits are internal and their edges are free of forces. A generalization to the case of slits emanating from the contour of the internal hole Λ will be made in Part 3.

Determination of the stress state of the plate reduces to finding the complex potentials $\Phi_v(z_v)$ ($v=1,2$) from the corresponding boundary conditions on the contours Λ, L [11]

$$a(t) \Phi_1^+(t_1) + b(t) \overline{\Phi_1^+(t_1)} + \Phi_2^+(t_2) = F(t), \quad t \in \Lambda \cup L_0, \quad (1.1)$$

$$a(t) \Phi_1^\pm(t_1) + b(t) \overline{\Phi_1^\pm(t_1)} + \Phi_2^\pm(t_2) = 0, \quad t \in L'. \quad (1.2)$$

$$a(t) = a_0 M_1(t)/M_2(t); \quad b(t) = b_0 \overline{M_1(t)}/M_2(t);$$

$$F(t) = \frac{X_n(t) + \bar{\mu}_2 Y_n(t)}{(\mu_2 - \bar{\mu}_2) M_2(t)}; \quad a_0 = (\mu_1 - \bar{\mu}_2)(\mu_2 - \bar{\mu}_2)^{-1};$$

$$b_0 = (\bar{\mu}_1 - \bar{\mu}_2)(\mu_2 - \bar{\mu}_2)^{-1}; \quad M_v(t) = \mu_v \cos \psi - \sin \psi \quad (v = 1, 2),$$

where the superscripts "+" and "-" denote the limiting value of the function as the contour is approached from the left (right); $\psi = \psi(t)$ is the angle between the normal $\bar{n}$ at the point t and the x axis; μ_v are roots of the corresponding characteristic equation.

We represent the functions $\Phi_V(z_V)$ in the form

$$\Phi_V(z_V) = \sum_{j=0}^1 \Phi_V^j(z_V). \quad (1.3)$$

Here, $\Phi_V^0(z_V)$ is the solution for an infinite anisotropic plate with an elliptical hole but without cracks. This solution satisfied boundary condition (1.1) on Λ . We will assume that $\Phi_V^0(z_V)$ is known [8].

Using the solution of the problem of the action of a single concentrated force at the point τ of an infinite anisotropic plate with an elliptical hole Λ free of external forces [2]

$$\Phi_V^*(z_V) = \frac{1}{\omega'_V(\xi_V)} \left[\frac{A_V}{\xi_V - \eta_V} + \frac{l_V \bar{A}_1}{\xi_V (\xi_V \bar{\eta}_1 - 1)} + \frac{n_V \bar{A}_2}{\xi_V (\xi_V \bar{\eta}_2 - 1)} \right]$$

and the superposition principle, we will seek $\Phi_V^1(z_V)$ in the following form

$$\begin{aligned} \Phi_V^1(z_V) &= \frac{1}{\omega'_V(\xi_V)} \int_L \left[\frac{A_V(\tau)}{\xi_V - \eta_V} + \frac{l_V \bar{A}_1(\tau)}{\xi_V (\xi_V \bar{\eta}_1 - 1)} + \frac{n_V \bar{A}_2(\tau)}{\xi_V (\xi_V \bar{\eta}_2 - 1)} \right] ds = \\ &= \frac{1}{2\pi i \omega'_V(\xi_V)} \int_L \left[\frac{\Omega_V(\tau) d\tau_V}{\eta_V - \xi_V} + \frac{l_V \bar{\Omega}_1(\tau) d\bar{\tau}_1}{\xi_V (\xi_V \bar{\eta}_1 - 1)} + \frac{n_V \bar{\Omega}_2(\tau) d\bar{\tau}_2}{\xi_V (\xi_V \bar{\eta}_2 - 1)} \right]. \\ \xi_V = \xi_V(z_V) &= (z_V + \sqrt{z_V^2 - a^2 - \mu_V^2 b^2}) (a - i\mu_V b)^{-1} (\xi_V(\infty) = \infty); \quad \eta_V = \xi_V(\tau_V); \\ z_V = \omega_V(\xi_V) &= \frac{1}{2} (a - i\mu_V b) \xi_V + \frac{1}{2} (a + i\mu_V b) \xi_V^{-1} (|\xi_V| > 1); \\ l_V &= (\mu_{3-V} - \bar{\mu}_1) (\mu_V - \mu_{3-V})^{-1}; \quad n_V = (\mu_{3-V} - \bar{\mu}_2) (\mu_V - \mu_{3-V})^{-1}; \\ \Omega_V(t) &= -2\pi i A_V(t) / M_V(t). \end{aligned} \quad (1.4)$$

where $\Omega_V(t) = \{\Omega_{Vj}(t)\}$ $t \in L_j$ ($j = \overline{0, n}$) are unknown complex functions on L . Here, we used the relation $d\tau_V = M_V(\tau) ds$ (ds is an element of the arc L).

Thus, the constructed functions $\Phi_V^1(z_V)$ satisfy the conditions $X_n = Y_n = 0$ on Λ . As a result, selection of $\Phi_V(z_V)$ in the form (1.3-1.4) automatically ensures satisfaction of boundary conditions (1.1) on Λ .

2. Inserting the limiting values of the function $\Phi_V(z_V)$ from (1.3-1.4) into boundary conditions (1.2), we obtain

$$a(t) \Phi_1(t_1) + b(t) \overline{\Phi_1(t_1)} + \Phi_2(t_2) = 0, \quad (2.1)$$

$$\Omega_2(t) = -a(t) \Omega_1(t) - b(t) \overline{\Omega_1(t)}, \quad t \in L'. \quad (2.2)$$

Following [11, 12], we will also assume that

$$\Omega_2(t) = -a(t) \Omega_1(t) + b(t) \overline{\Omega_1(t)}, \quad t \in L_0. \quad (2.3)$$

With allowance for (1.3), (1.4), (2.2), and (2.3), after certain transformations we can use boundary conditions (1.1) and (2.1) to obtain a mixed system of singular (on L') and regular (on L_0) integral equations relative to the sought functions $\Omega_1(t)$

$$2\delta_{0k}b(t)\Omega_1(t) + \int_L [K_1(t, \tau)\Omega_1(\tau) + K_2(t, \tau)\overline{\Omega_1(\tau)}] ds = f_k(t), \quad t \in L, \quad k = \overline{0, n}.$$

$$\begin{aligned} K_1(t, \tau) ds &= \frac{a(t)}{\pi i \omega'_1(\xi_1)} \left[\frac{d\tau_1}{\eta_1 - \xi_1} + \frac{(-1)^j n_1 \overline{b(\tau)} d\tau_2}{\xi_1(1 - \xi_1 \eta_2)} \right] + \frac{b(t)}{\pi i \omega'_1(\xi_1)} \times \\ &\times \left[\frac{\overline{l_1} d\tau_1}{\xi_1(1 - \xi_1 \eta_1)} - \frac{\overline{n_1} a(\tau) d\tau_2}{\xi_1(1 - \xi_1 \eta_2)} \right] - \frac{1}{\pi i \omega'_2(\xi_2)} \times \\ &\times \left[\frac{a(\tau) d\tau_2}{\eta_2 - \xi_2} - \frac{(-1)^j n_2 \overline{b(\tau)} d\tau_2}{\xi_2(1 - \xi_2 \eta_2)} \right]; \\ K_2(t, \tau) ds &= \frac{a(t)}{\pi i \omega'_1(\xi_1)} \left[\frac{l_1 d\tau_1}{\xi_1(\xi_1 \eta_1 - 1)} + \frac{n_1 a(\tau) d\tau_2}{\xi_1(1 - \xi_1 \eta_2)} \right] \times \frac{b(t)}{\pi i \omega'_1(\xi_1)} \times \\ &\times \left[\frac{d\tau_1}{\eta_1 - \xi_1} + \frac{(-1)^j \overline{n_1} b(\tau) d\tau_2}{\xi_1(1 - \xi_1 \eta_2)} \right] - \frac{1}{\pi i \omega'_2(\xi_2)} \left[\frac{(-1)^j b(\tau) d\tau_2}{\eta_2 - \xi_2} + \right. \\ &\left. + \frac{l_2 d\tau_1}{\xi_2(1 - \xi_2 \eta_1)} - \frac{n_2 \overline{a(\tau)} d\tau_2}{\xi_2(1 - \xi_2 \eta_2)} \right]; \\ f_0(t) &= 2F(t) - 2[a(t)\Phi_1^0(t_1) - b(t)\Phi_1^0(t_1) + \Phi_2^0(t_2)], \quad t \in L_0; \\ f_k(t) &= -2[a(t)\Phi_1^0(t_1) + b(t)\overline{\Phi_1^0(t_1)} + \Phi_2^0(t_2)] \quad t \in L', \quad (k = \overline{1, n}). \end{aligned} \quad (2.4)$$

Here, δ_{0k} is the Kronecker symbol; $j = 0(1)$ with $t \in L'(L_0)$.

Along with the condition of nonambiguity of the displacements in the circumvention of each slit L_j ($j = \overline{1, n}$), Eqs. (2.4) are augmented by the conditions:

$$\int_{L_j} \Omega_{1j}(\tau) d\tau_1 = 0. \quad (2.5)$$

3. As was done in [12], we can show here that the system of integral equations (2.4), together with auxiliary conditions (2.5), is uniquely solvable. Here, $\Omega_1(t)$ satisfies Holder's inequality on L_0 and belongs to the class H^* on L' with the index 0.5 [6].

Using the Gauss-Chebyshev formulas for the integrals over $L_j = \{t = \tau^j(\xi) \mid |\xi| < 1\}$ ($j = \overline{1, n}$), and the rectangle formulas for the integrals over $L_0 = \{t = \tau^0(\xi) \mid 0 < \xi < T(\tau_0(0) = \tau^0(T))\}$, we reduced the solution of system (2.4) with conditions (2.5) to the solution of a system of linear algebraic equations relative to approximate values of the functions $\chi_j(\xi) = \Omega_{1j}[\tau^j(\xi)] \times \sqrt{1 - \xi^2}$, and the Chebyshev nodes $\xi_{ij} = \cos((2i-1)/2N_j)$ ($i = \overline{1, N_j}$; N_j is the number of nodes on the contour L_j) $\chi_0(\xi) = \Omega_{10}[\tau^0(\xi)]$ and at arbitrarily chosen points ξ_i ($0 < \xi_1 < \xi_2 < \dots < L_j$) $< \xi_{N_0} < T$; N_0 is the number of nodes on the contour L_0). Theoretical estimates of the convergence of the numerical method of solution being used here can be found in [1, 3], for example.

If system (2.4-2.5) is solvable and if values of $\chi_j(\pm 1)$ ($j = \overline{1, n}$) have been found, then we can use the asymptotic formulas from [5] to find the stress distribution and the SIF for rupture and shear

$$k_1(c) = \lim_{t \rightarrow c} \sigma_n \sqrt{2\pi r}; \quad k_2(c) = \lim_{t \rightarrow c} \tau_n \sqrt{2\pi r}$$

at the tips $c = \tau^j(\mp 1)$ of the crack L_j . Here, t is a point lying on an extension of the cracks beyond the tip c along a tangent; $r = |t - c|$.

If the end a_p of the slit L_p reaches the contour of the hole Λ , then condition (2.5) can be ignored for $j = p$. Following [9], we need to refine the corresponding potential in (1.4) and integral equations (2.4) by an established method. To numerically solve system (2.4), we will make use of a simplified procedure based on the above-described algorithm for internal slits.

The potentials (1.4) turn out to be an effective means of determining the stresses in the neighborhood of crack tips in finite anisotropic plates with an elliptical hole.

Below, we present results of calculations of the rupture SIF k_1 for problems involving circular ($L_0 = \{|z_0| = R_0\}$) and rectangular $L_0 = \{x = \pm W; y = \pm 2W\}$ plates (cases a and b, respectively) made of orthotropic materials with one or two radial edge cracks $L_{1,2} = \{t = \tau_{1,2}(\xi) = \pm[a + l(1 + \xi)] \mid |\xi| < 1\}$. The angle φ formed by the principal direction of anisotropy E_1 and the x axis is assumed to be equal to 0 or $\pi/2$. The anisotropy parameters of the materials are as follows at $\varphi = 0$ [4]: 1) $E_1/E_2 = 1$; $\mu_1 = 1.01$; $\mu_2 = 0.99$ (nearly isotropic material) 2) $E_1/E_2 = 2$; $\mu_1 = 4.11$; $\mu_2 = 0.343$; 3) $E_1/E_2 = 2.65$; $\mu_1 = 1.679$; $\mu_2 = 0.971$; 4) $E_1/E_2 = 23.8$; $\mu_1 = 3.26$; $\mu_2 = 1.50$. The plate is tensioned by uniformly distributed normal forces p applied to the contour L_0 (case a) or to the edges $y = \pm 2W$ (case b); the edges $x = \pm W$ are free of external forces. The contour Λ is unloaded in both cases.

Considering the symmetry of the problem relative to the x axis (in the case of one crack) or the axes x and y (in the case of two cracks), the number of linear algebraic equations to which the system of integral equations can be reduced can be decreased by a factor of two or four, respectively.

Figures 2 and 3 and Tables 1 and 2 show for case (a) the results of study of the effect of crack length on k_1 with different degrees of compression of the ellipse ($\varepsilon = b/a$).

Figure 2 shows curves depicting the dependence of the dimensionless SIF for rupture $K = k_1/p\sqrt{2\pi l}$ on the parameter $\lambda = 2l/(R_0 - a)$ with $\varepsilon = 1$, $a/R_0 = 0.5$ for materials 1-3 (curve 1-3) in the case of one crack. Here and below, the solid (dashed lines) correspond to $\varphi = 0$ ($\pi/2$) (for material 1, the values of K calculated with $\varphi = 0$ and $\pi/2$ nearly coincide and are given with $\varphi = 0$). The corresponding numerical values of K for material 1 with different values of λ are shown in Table 1 for the case of one (curve 1, Fig. 2) and two cracks. Also shown for comparison is well-known data for an isotropic material [10, 13, 15, 16]. Stable solutions were obtained in the range $0 < \lambda \leq 0.7$ with allowance for the symmetry of the problem in the case of one crack with the number of nodes on the semicircle $N_0/2 = 60$ and in the case of two cracks with the number of nodes on one-fourth of the circle $N_0/4 = 20$. Here and below, we used a uniform subdivision on the external contour, while the number of nodes on the cracks $N_{1,2} = 20$. At $\lambda > 0.7$, to find K is necessary to increase the number of nodes to 80 on the semicircle and to 30 on the quarter-circle. With a further increase in the number of nodes on the external contour, the results differ from one another by less than 1%.

The presence of a hole has a significant effect on the value of K for short cracks ($\lambda \leq 0.1$). As the crack approaches the external contour, there is a substantial increase in K . An increase in the degree of anisotropy E_1/E_2 is accompanied by an increase in the difference in the values of K calculated with $\varphi = 0$ and $\pi/2$. Comparison of the numerical data for material 1 with the well-known solutions for an isotropic material [10, 13] (Table 1) shows their good agreement. The maximum relative error of the sets of data being compared is on the order of 2%. The result obtained in [15] with $\lambda = 0.8$ should be considered understated.

Figure 3 shows the dependence of the correction SIF for rupture $K = k_1/k^*_1$ on λ at $a/R_0 = 0.5$ and $\varepsilon = 0; 0.2; 0.5; 1$ (curves 1-4). The dependence shown is for material 3 in the case of two cracks. Here k^*_1 is the rupture SIF at the tips of the cracks at $\varepsilon = 0$. Table 2 shows dimensionless numerical values of $k^*_1/p\sqrt{\pi(2l+a)}$ (the problem of a circular disk with an "equivalent" central crack $L^*_1 = \{|x| < a + 2l; y = 0\}$) in relation to $\lambda^* = (2l+a)/R_0$ for materials 1-4. Here, for comparison the denominator shows data from [4]. Stable solutions (accurate to within 1%) were obtained with the same number of nodes on the external contour ($N_0/4 = 20$ for $0 < \lambda \leq 0.7$; $N_0/4 = 30$ for $\lambda > 0.7$), as in the previous example in the case of two cracks. The results obtained show that for short cracks ($\lambda < 0.1$) the value of K depends appreciably on the degree of compression of the ellipse. Even at small values of ε , it deviates considerably from 1. An increase in the compression of the ellipse (a decrease in ε) is accompanied by a decrease in the difference in the values of K calculated with $\varphi = 0$ and $\pi/2$ (they coincide at $\varepsilon = 0$). The hole can be regarded as a defect of the crack-slit type when $\varepsilon \leq 0.2$, $\lambda \geq 0.1$, $\varphi = 0$ and $\pi/2$. Comparison of the numerical data shown in Table 2 illustrates their good agreement within the range $0 < \lambda \leq 0.9$. Here, the maximum relative difference between the results being compared is on the order of 1% except for the data for material 4 at $\varphi = 0$, $\lambda = 0.8$ (2%) and $\lambda = 0.9$ (12%).

For case (b), Fig. 4 shows curves depicting the dependence of $K = k_1/k^*_1$ on $\lambda = (2l+a)/W$ with $\varepsilon = 1$, $a/W = 0.1; 0.25; 0.5$ (curves 1-3). These results are for material 2 in the case of two cracks. Here, $k^*_1 = p\sqrt{\pi(2l+a)}$ is the rupture SIF at the tips of an isolated "equivalent" crack L^*_1 in a region loaded at infinity by the forces p . Stable solutions (accurate to within 1%) were obtained with the number of nodes on the external contour $N_0 = 80$ in the range $0 < \lambda \leq 0.6$; for $\lambda > 0.6$, the number of nodes must be doubled. It follows from the figure that

TABLE 1

λ	One crack				Two cracks		
	Present study	[10]	[13]	[15]	Present study	[10]	[16]
0,1	2,79	2,78	2,81	2,72	2,84	2,84	2,8
0,2	2,78	2,78	2,78	2,79	2,96	2,96	2,9
0,3	2,85	2,85	2,87	2,86	3,21	3,21	3,2
0,4	2,96	2,97	2,99	3,01	3,55	3,55	3,4
0,5	3,11	3,12	3,14	3,14	3,98	3,98	3,9
0,6	3,30	3,31	3,32	3,30	4,49	4,50	4,4
0,7	3,55	3,61	3,55	3,48	5,09	5,20	5,0
0,8	3,95	3,97	3,91	3,73	5,82	5,86	5,7

TABLE 2

Material	φ	λ^*					
		0,1	0,5	0,6	0,7	0,8	0,9
1	0	1,015	1,387	1,580	1,840	2,237	3,036
		1,015	1,387	1,578	1,840	2,238	3,038
2	0	1,022	1,410	1,574	1,796	2,139	2,849
		1,022	1,411	1,575	1,798	2,137	2,823
	$\pi/2$	1,042	1,625	1,849	2,143	2,578	3,395
		1,042	1,627	1,852	2,147	2,583	3,385
3	0	1,010	1,284	1,436	1,658	2,018	2,796
		1,010	1,284	1,437	1,658	2,020	2,799
	$\pi/2$	1,023	1,559	1,819	2,161	2,641	3,488
		1,023	1,560	1,820	2,163	2,644	3,492
4	0	1,007	1,201	1,323	1,513	1,844	2,601
		1,006	1,202	1,324	1,514	1,877	2,313
	$\pi/2$	1,062	2,211	2,740	3,464	4,524	6,178
		1,062	2,218	2,753	3,488	4,567	6,248

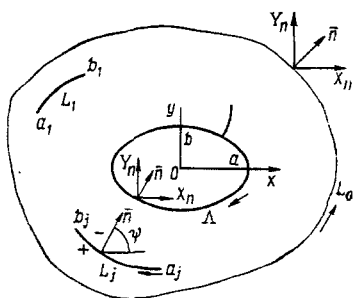


Fig. 1

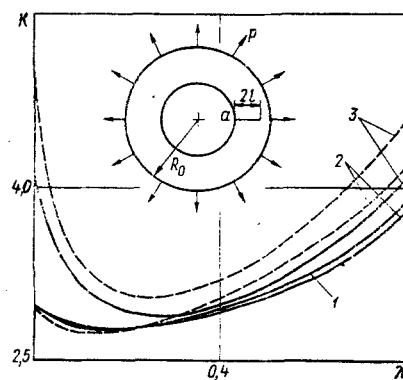


Fig. 2

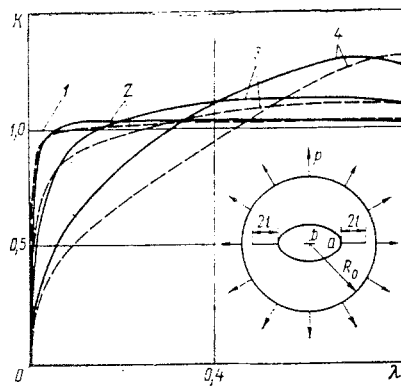


Fig. 3

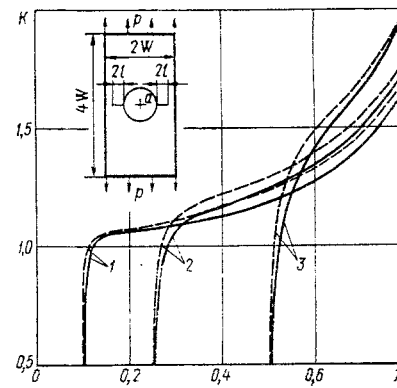


Fig. 4

even for broad regions ($a/W = 0.1$), the hole cannot be regarded as a defect of the type represented by an isolated crack-slit in an infinite region. The angle φ has a negligible effect on the value of K in the investigated parameter range. It should be noted that the values of K calculated for material 1 agree within 2% with the data in [14].

The good agreement of the compared numerical solutions (Tables 1 and 2) and the stability of the calculation make it possible to consider the results obtained reliable.

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