

STRESSED STATE OF A HALF PLANE WITH SHALLOW EDGE CRACK UNDER HERTZIAN LOADING (A SURVEY)

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We present a survey of works dealing with the stress intensity factors (SIF) for shallow edge cracks formed in elastic half planes subjected to the action of Hertzian loads moving along the boundaries of the planes. The numerical data on the SIF obtained by the method of singular integral equations are compared with the results of the other authors. The best agreement is observed with the results obtained by Keer and Bower by using the dislocation approach. The agreement with the results obtained by the finite-element method is satisfactory.

The surface cracks often formed in rolling bodies of revolution first grow at characteristic small angles of $10\text{--}30^\circ$ to the surface of rolling [1–4]. Under laboratory conditions, these cracks were first obtained by Way in 1935 who rolled a steel disk with the help of the second disk under loading [5]. Since that time, special experimental and theoretical investigations of the initiation and growth of cracks in rolling bodies were originated (see the surveys in [1, 2, 4, 6–10]).

In the theoretical investigations, significant attention is given to the numerical analysis of the stress intensity factors (SIF) for surface cracks formed in rolling bodies of revolution regarded as basic parameters of fracture of the rolling surfaces used for the assessment of their contact durability. For the evaluation of these parameters, it is customary to use various model problems and methods. The contact action of the rider is, as a rule, modeled by Hertzian contact forces with tangential component. The damaged rolling body is modeled either as an elastic half plane weakened by a rectilinear edge crack or as a half space containing a surface (semicircular or semielliptic) crack. The conditions of dry friction between the contacting bodies are simulated by large values of the friction coefficient, whereas the conditions of wetting (lubrication) are simulated by low values of the indicated coefficient. It is assumed that the liquid penetrating into the crack either wedges the crack lips or only decreases the level of friction between the lips.

The first theoretical investigations in this field were carried out by Keer and his colleagues for the two-dimensional model problem [first, for a simpler case of vertical crack [11] (1982) and then for the oblique crack [1] (1983)]. Somewhat later, in 1985, Murakami, Kaneta, and Yatsuzuka [12] used a three-dimensional model for the evaluation of the SIF in rolling solids. Actually, they studied the case of a half space weakened by a semicircular surface crack subjected to the action of Hertzian loading (in the process of rolling of a circular cylinder with sliding). The case of a semielliptic surface crack was analyzed by Kaneta and Murakami in [13].

Note that different authors use different methods for the solution of two- and three-dimensional problems. In the two-dimensional case, Keer with colleagues reduced the analyzed problem to the solution of singular integral equations (SIE) by using the dislocation approach [1, 11, 14]. The same approach was also used by Bower [15] and other authors [16, 17]. The problem of a half plane weakened by a shallow edge crack subjected to the action of Hertzian loading was also solved by the methods of boundary integral equations [18], boundary elements [19], and weight functions [20]. The authors of the present work solved the indicated problem by the method of SIE [10, 21–26]. At present, the finite-element method is extensively used for the solution of this

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problem [3, 27–30]. In the three-dimensional case, the researchers mainly use the method of bulk forces [2, 3, 31] and the finite-element method [32].

All cited works deal with the investigation of the influence of various operating factors (such as the friction forces between the rolling bodies and crack lips, the length and orientation of the crack, the size of the contact zone of the rolling bodies, and the pressure of oil in the crack) on the behavior of the SIF in the process of motion of contact loading along the edge of the half plane.

Note that the two-dimensional model problem of an elastic half plane with edge crack subjected to the action of a moving Hertzian load is extensively used in modeling the contact rolling interaction and fracture processes in rolling bodies. Therefore, in what follows, we analyze and compare the numerical results obtained for the SIF by different researchers. We study various versions of the boundary conditions characterizing the interaction of crack lips under the conditions of rolling contact, including the presence of lubricants.

The complexity of the solution of problems of this sort is connected with the fact that, in some regions, the crack subjected to the action of loading moving along the edge of a half plane can be either closed (in this case, both sliding and adhesion of the crack lips are possible) or open. Thus, we have a complex contact problem with mixed boundary conditions. Moreover, as the angle made by the crack with the edge of the plane decreases, the accuracy of numerical calculations becomes much lower. This is why it is reasonable to ask how exact are the values of SIF obtained by using the above-mentioned methods for shallow edge cracks in the half plane subjected to the action of the Hertzian load.

Statement of the Problem

Assume that one of the bodies in a rolling couple (driven body) is damaged by a surface macrocrack. Instead of the actual body, we consider an elastic half plane with edge notch (crack) in the two-dimensional statement of the problem. The contact influence of the other body (rider) upon the analyzed body is simulated by a unidirectional repeated translational motion (along the edge of the half plane) of Hertzian contact forces with the following tangential component (Fig. 1):

$$s(x) = -\frac{p_0(1 + if_s)\sqrt{a^2 - (x - x_0)^2}}{a}, \quad |x - x_0| \leq a, \quad (1)$$

where f_s is the coefficient of contact friction between the rolling bodies. The location of the contact zone relative to the crack mouth is specified by the parameter $\lambda = x_0/l$, the relative crack length is specified by the parameter $\varepsilon = l/a$, and the orientation of the crack is specified by the angle β . Note that, for the given direction of motion of the rider (contact load), the tangential forces acting upon the surface of the driven body have the opposite direction (along the Ox -axis) and $f_s > 0$. For the opposite (left–right) direction of motion of the rider, we have $f_s < 0$.

If the contact load moves along the edge of the half plane, e.g., in the right–left direction (Fig. 1), then there are three possible types of interaction between the crack lips: the crack is open if the load is applied at a sufficiently large distance to the right of the mouth and the crack partially or completely closed if the load moves over its mouth and further. This is why, in general, we can use the following boundary conditions of the problem:

$$\sigma_y(x) - i\tau_{xy}(x) = \begin{cases} s(x), & |x - x_0| \leq a, \quad y = 0, \\ 0, & |x - x_0| > a, \quad y = 0, \end{cases} \quad (2)$$

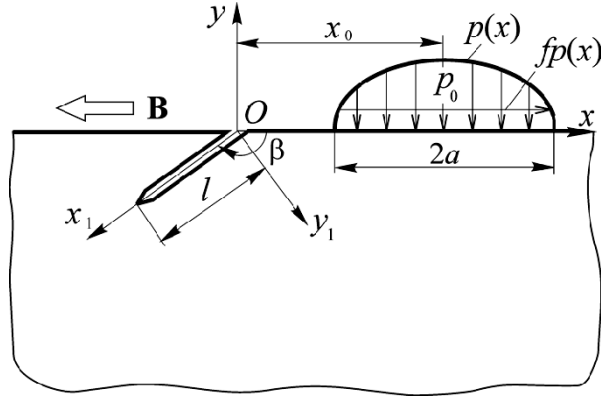


Fig. 1. Schematic diagram of the problem (geometry and forces); **B** direction of motion of the contact load.

on the edge of the half plane and

$$N^{\pm}(x_1) + iT^{\pm}(x_1) = 0, \quad x_1 \in L_{op}, \quad (3)$$

$$v^{+}(x_1) - v^{-}(x_1) = 0, \quad x_1 \in L_{sl} \cup L_{st}, \quad (4)$$

$$T^{\pm}(x_1) = f_c \operatorname{sgn}[T^{\pm}(x_1)] |N(x_1)|, \quad |T(x_1)| > f_c |N(x_1)|, \quad x_1 \in L_{sl}, \quad (5)$$

$$\frac{d}{dx_1} [u^{+}(x_1) - u^{-}(x_1)] = 0, \quad |T(x_1)| < f_c |N(x_1)|, \quad x_1 \in L_{st}, \quad (6)$$

on the crack lips. In this case, the crack contour L is formed by a collection of zones L_{op} (in which the crack is open), L_{sl} (in which the crack lips are sliding), and L_{st} (in which the lips are coupled). The coefficient of friction between the crack lips is denoted by f_c and σ_y and τ_{xy} , N , and T , v and u are, respectively the normal and tangential components of forces acting on the edge of the plane, the forces acting on the crack lips, and the displacements of the crack lips. The superscripts “+” and “-” denote the limiting values of the corresponding quantities on approaching the crack contour from the left and from the right, respectively.

We reduce the problem to the solution of SIE for the unknown derivatives of discontinuities of displacements along the crack with their subsequent reduction to the system of linear algebraic equations by the Gauss–Chebyshev method of mechanical quadratures. The SIF are determined as a result of the solution of this system.

First Basic Problem

We consider the cases where the crack lips are not contact in the process motion of the load. In these cases, the crack opens either as a result of action of the tangential forces formed in the process of sliding of the rider (dry friction) or under the action of pressure of the liquid entrapped under loading upon the crack lips (conditions of boundary lubrication).

Crack with Free Lips. In a simple case where the contact load is applied at a sufficiently large distance to the right of the crack mouth in the process of its motion in the right–left direction and the crack is open, the boundary conditions of the problem take the form (2), (3). In this case, it is clear that $L_{op} = L$.

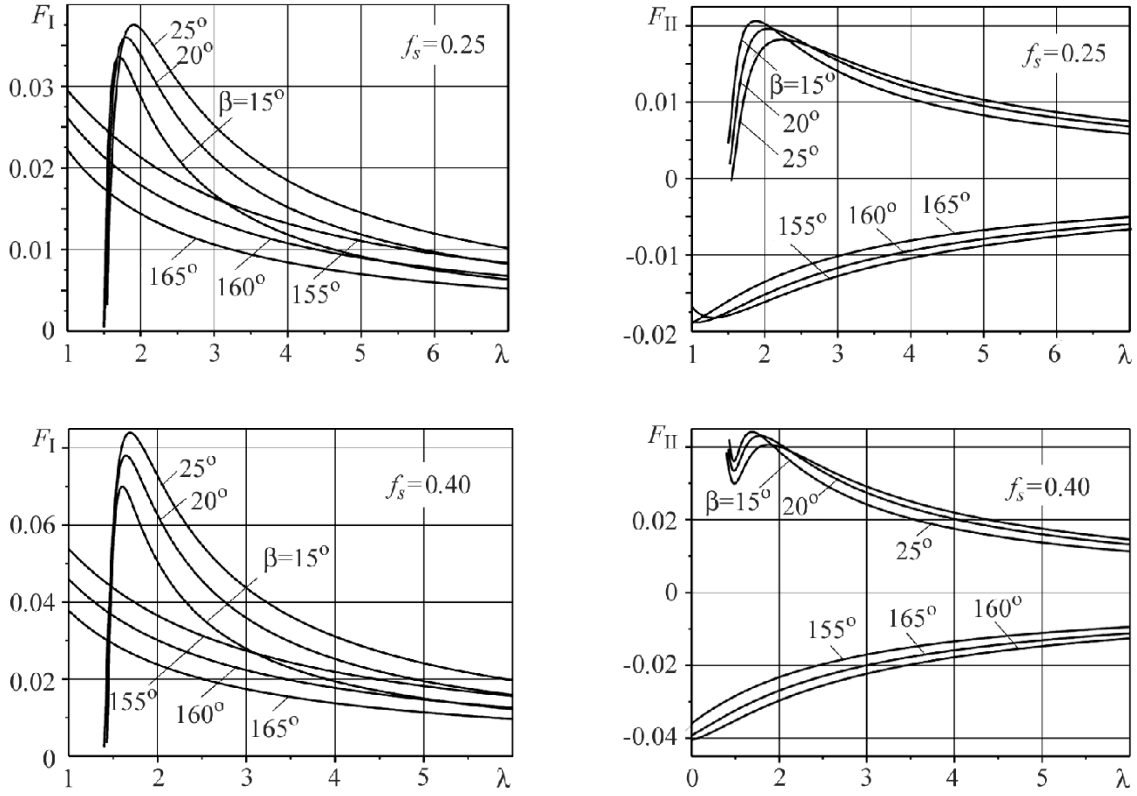


Fig. 2. Dependences of the dimensionless SIF $F_{I,II} = K_{I,II} / (p_0 \sqrt{\pi a})$ for an open arbitrarily oriented crack on the location of the contact load; $\varepsilon = l/a = 0.5$.

The influence of the orientation of the crack on the SIF is studied for two values of the coefficients of friction between the rolling bodies $f_c = 0.25$ and 0.40 . Specifically, we establish the ranges of the parameter λ in which the crack is open. For a given direction of action of the tangential forces τ_{xy} , the corresponding ranges of λ are located to the right of the crack mouth ($\lambda \geq 1$). For the other values of λ , the crack lips are in complete or partial contact.

As expected, for the open crack, the factors K_I and K_{II} rapidly vary as the contact load moves near the crack mouth (Fig. 2). In this case, the SIF attain their maximum absolute values (increasing with the value of the friction coefficient f_c). Moreover, the SIF gradually vanish as the distance between the contact area and the crack mouth increases. The maximum values of the SIF for the acute angles of orientation of the crack are larger than for the blunt angles. Moreover, these values increase with the angle made by the oblique crack with the edge of the plate.

The influence of the length of the contact area on the SIF is studied in [21]. The relative length of the contact area is specified by the parameter $\varepsilon_0 = 1/\varepsilon = a/l$. For $\varepsilon_0 = 0$, we, clearly, get a special case of concentrated forces [33]. On the basis of the analysis of the plots presented in Fig. 3), it is possible to conclude that the behaviors of the curves of K_I and K_{II} for the Hertzian load and concentrated forces are similar. However, the concentrated force induces the largest maximum values of K_I and K_{II} decreasing as the length of the contact area increases.

The comparison of the values of the SIF for vertical cracks available from the literature [11] with our data reveals their good correlation (Table 1). Here and in what follows, the tables contain only the maximum values of the SIF responsible for the fracture of contacting bodies.

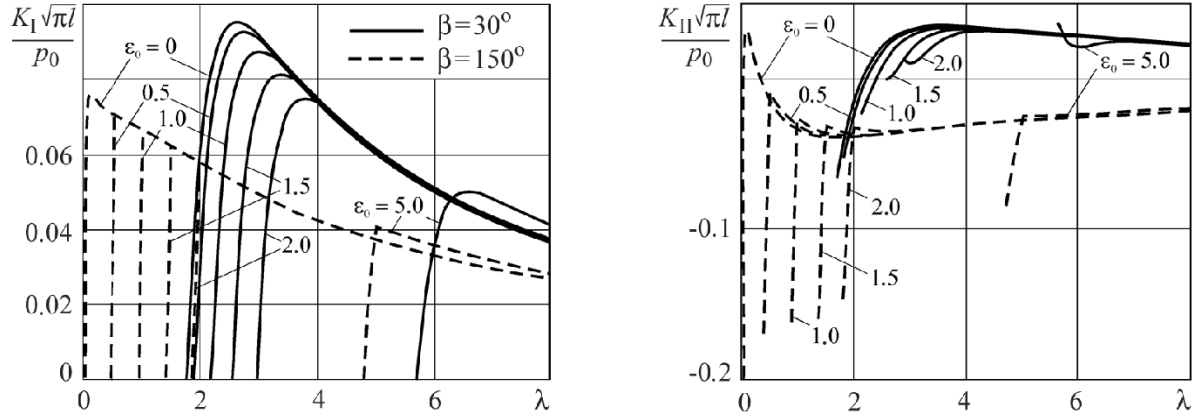


Fig. 3. Dependences of the dimensionless SIF $K_{I,II}\sqrt{\pi l}/p_0$ for the open crack on the location of loading for various relative lengths of the contact area; $f_c = 0.3$ [21].

Table 1. Comparison of the Maximum Values of Normalized SIF $K_{I,II}/(p_0\sqrt{\pi a})$ Available from the Literature [11] with the Values Obtained in the Present Work; $\beta = 90^\circ$

ϵ	0.5	1.0	2.0
$f_c = 0.25$			
[11]	0.04	0.03	0.02
Present work	0.0384 ($\lambda = 2.72$)	0.0278 ($\lambda = 5.06$)	0.0198 ($\lambda = 9.91$)
$f_c = 0.5$			
[11]	0.13	0.03	0.02
Present work	0.1357 ($\lambda = 1.65$)	0.1026 ($\lambda = 2.74$)	0.0740 ($\lambda = 5.17$)

We also compare our data with the SIF obtained by Benuzzi, Bormetti, and Donzella for an oblique open crack by the finite-element method [30] (Fig. 4). It is easy to see that the values of K_I almost coincide but the values of the factor K_{II} are characterized by noticeable disagreement. In our opinion, the values of K_{II} obtained in [30] are erroneous.

We also compare our results with the data obtained by Ringsberg and Bergkvist [3] (Table 2) and Bogdanski and Trajer [29] (Table 3) for the open crack (with $\lambda = -1$ when the values of the quantity $K_I/(p_0\sqrt{a})$ are maximum) by the finite-element method as in [30]. From the data presented in the tables, we see that the results obtained by Ringsberg and Bergkvist are closer to our results.

Thus, the comparative analysis of the SIF for the open crack shows that our results coincide with the data obtained by Keer et al. by using the dislocation approach [11]. They are in good agreement with the SIF obtained by the finite-element method (Table 2) [3].

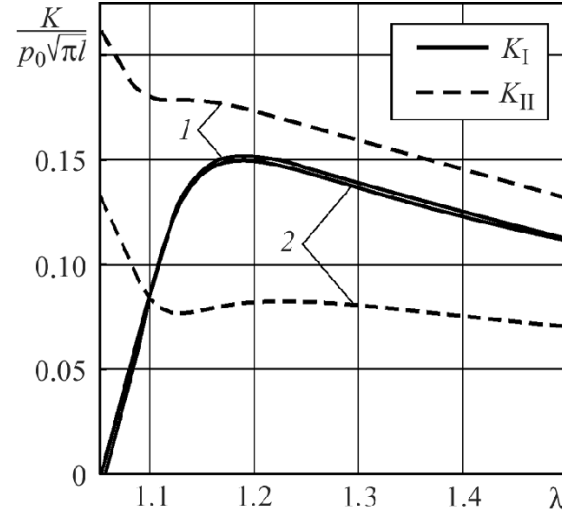


Fig. 4. Comparison of the dimensionless SIF $K_{I,II} / (p_0 \sqrt{\pi} l)$ for the open crack obtained according to the data presented in [30] (curves 1) and according to our data (curves 2); $f_c = 0.3$, $\beta = 30^\circ$, and $\varepsilon = 0.125$.

Table 2. Comparison of the Data Presented in [30] with Our Data for the SIF; $f_s = -0.4$, $\varepsilon = 0.5$, and $\lambda = -1$

β	10°	15°	20°
$K_I / (p_0 \sqrt{a})$			
[30]	0.060	0.080	0.098
Present work	0.053	0.067	0.081
$K_{II} / (p_0 \sqrt{a})$			
[30]	0.060	0.079	0.086
Present work	0.056	0.063	0.070

The disagreement observed in this case can be explained by the specific features of the finite-element method in which a rectangle of finite sizes cut out from the model semiinfinite region is analyzed instead of this region. The data presented in Table 3 enable us to assume that the values established in [29] are incorrect.

Pressure on the Crack Lips (Influence of Lubricants). Lubricating liquids (oil or water) can strongly affect the resistance of bodies to surface fracture under the conditions of contact rolling fatigue. Bower [15] studied three possible effects of penetration of the liquid into the crack:

- (a) liquid lubricates the crack lips and decreases friction between the lips;

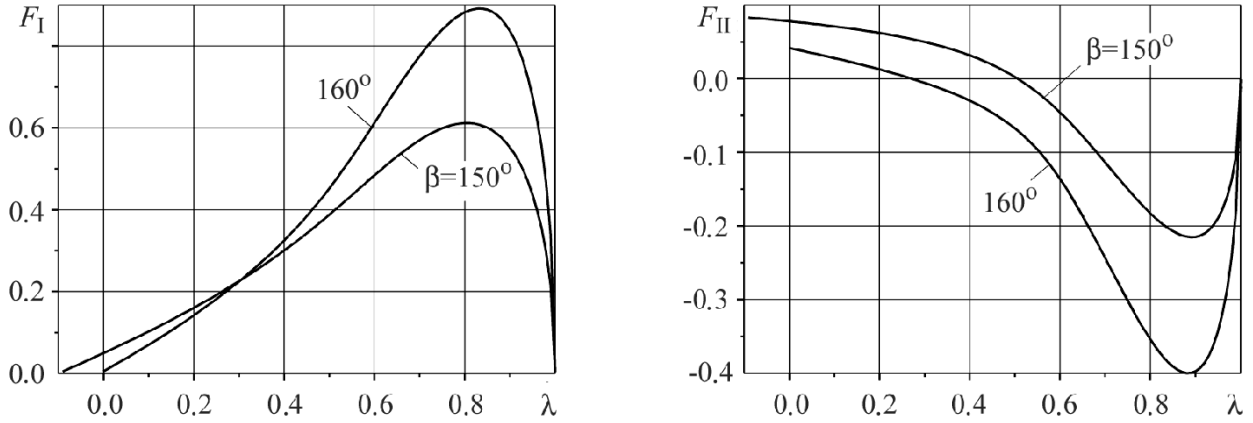


Fig. 5. Dependences of the dimensionless SIF $F_{I,II} = K_{I,II} / (p_0 \sqrt{\pi l})$ on the location of load over the crack mouth in the presence of lubricants; $f_s = 0.1$ and $\varepsilon = 0.5$ [26].

Table 3. Comparison of the Data Taken from [29] with our Data on the Values of the SIF;
 $f_s = 0.1$, $\beta = 15^\circ$, and $\varepsilon = 0.5$; $\lambda = -1$

Reference	$K_I / (p_0 \sqrt{a})$	$K_{II} / (p_0 \sqrt{a})$
[29]	0.035	0.050
Present work	0.012	0.0030

(b) liquid transfers contact pressure acting upon the lips of the open crack;

(c) liquid is entrapped by the crack and the pressure forces act upon the lips in a certain region in the vicinity of the crack tip.

It is worth noting that the mechanisms of crack propagation (b) and (c) were earlier discussed by Keer and Bryant [1] and Kaneta and Murakami [34].

We now consider case (b) in more detail. In this case, as the rider covers the crack mouth in a loading cycle, the liquid begins to press upon the crack lips and wedges them. As the rider leaves the crack mouth, the pressure acting upon the lips vanishes. The major part of the authors [15, 23, 26, 27, 34, 35] believes that the lips of the crack whose mouth is covered by the rider ($|\lambda| \leq 1$) suffer the action of constant pressure equal to the pressure in the crack mouth under contact loading. As the rider passes the crack mouth ($|\lambda| > 1$), conditions (3)–(6) are satisfied.

In the work by Panasyuk, Datsyshyn, and Glazov [26], one can find the data on the influence of the orientation of the crack on the SIF. As the angle made by the crack with the edge of the plane increases, the maximum absolute values of the SIF decrease as shown for $f_s = 0.1$ and $\varepsilon = 0.5$ (Fig. 5). Note that these values are attained when the contact load begins to cover the crack mouth in moving in the right–left direction (Fig. 1). The comparison of the values of SIF obtained for $f_s = -0.05$, $\beta = 25^\circ$, and $\varepsilon = 0.5$ by Bower [15] with the data presented [26] reveals their good correlation (Fig. 6).

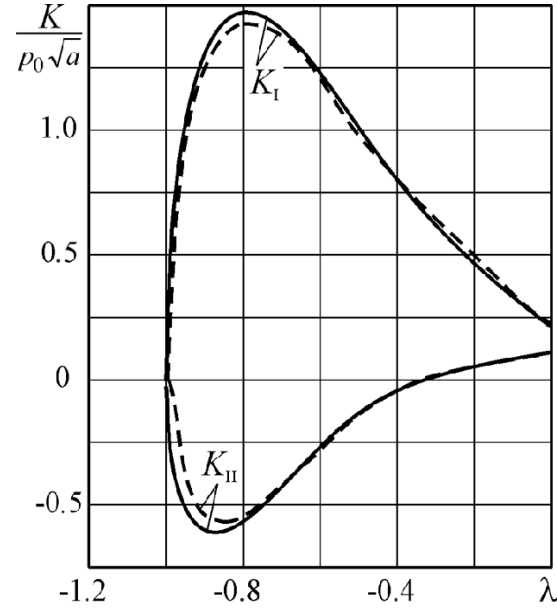
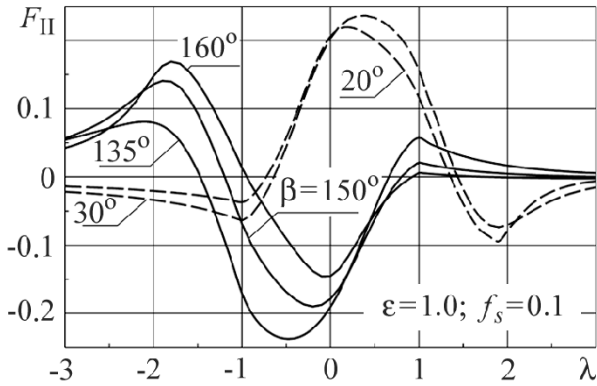
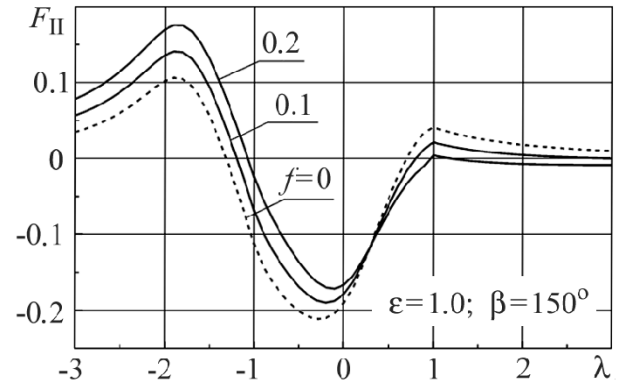


Fig. 6. Comparison of the dimensionless SIF $K_{I,II} / (p_0 \sqrt{a})$ in the presence of lubricants obtained by Bower [15] (dashed lines) with the results presented in [26] (solid lines); $f_s = -0.05$, $\beta = 25^\circ$, and $\epsilon = 0.5$.



(a)



(b)

Fig. 7. Dependences of the dimensionless SIF $F_{II} = K_{II} / (p_0 \sqrt{\pi a})$ on the location of the load for $f_s = 0$: (a) for different orientations of the cracks β , (b) for different friction coefficients f_s [24].

Mixed Problem (Contact of the Crack Lips)

As already indicated, for a certain location of the contact load in rolling cycles (for some ranges of the parameter λ), the crack lips are in contact.

Smooth Contact of the Crack Lips. For a special case of smooth contact of the crack lips ($f_s = 0$), the boundary conditions of the problem take the form (2) on the edge of the half plane. Moreover, on the crack lips, we get the following conditions:

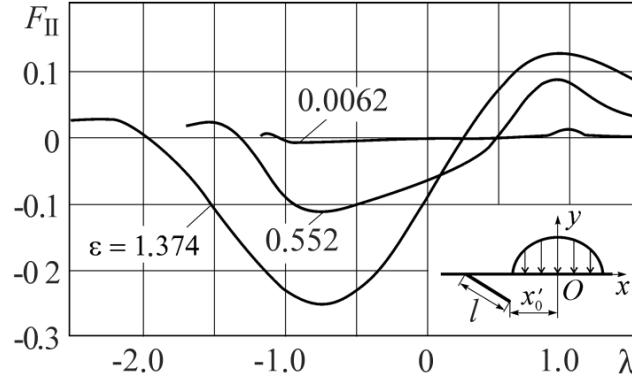


Fig. 8. Dependences of the dimensionless SIF $F_{II} = K_{II}/(p_0\sqrt{\pi a})$ on the relative distance of the load from the crack tip $\lambda' = x'_0/l$; $f_c = f_s = 0$, $\beta = 25^\circ$ [1].

$$N^\pm(x_1) = 0, \quad x_1 \in L_{op}, \quad (7)$$

$$T^\pm(x_1) = 0, \quad x_1 \in L, \quad (8)$$

$$v^+(x_1) - v^-(x_1) = 0, \quad x_1 \in L_{sl}. \quad (9)$$

The influence of the orientation of the crack and friction forces acting upon the rolling bodies on the SIF of a crack with smooth contact of the lips is investigated in [24]. For different orientations of the crack, the factor K_{II} varies as a function of the parameter λ and takes its maximum and minimum values for the corresponding values of λ (Fig. 7a). The analysis of the numerical data shows that the values of $\lambda = \lambda^*$ guaranteeing the maximum values of $|K_{II}|$ correspond to the crack closed in the vicinity of the tip. Thus, for $\beta = 150^\circ$, we have $\lambda^* \approx -0.2$ and, for $\beta = 30^\circ$, $\lambda^* \approx 0.3$. As soon as the contact load completely passes through the crack mouth, the quantity K_{II} changes its sign. On the other hand, the variations of the friction coefficient can strongly affect the location of the contact load guaranteeing the attainment of the maximum value of $|K_{II}|$ (Fig. 7b). Specifically, for a crack with the angle of orientation $\beta = 150^\circ$, the required extremum is attained even for $f_s = 0.1$ as a local minimum ($\lambda^* \approx -0.2$). For $f_s = 0.2$, this extremum turns into a local maximum ($\lambda^* \approx -1.9$).

In [1], Keer and Bryant studied, in particular, the case of smooth contact of the crack lips along the entire crack length ($f_c = f_s = 0$). For various relative crack lengths with the angle of orientation $\beta = 25^\circ$, we plotted the dependences of the dimensionless SIF F_{II} on the relative distance of the crack tip from the center of the Hertzian contact zone

$$\lambda' = \frac{x'_0}{l} = -\lambda + \varepsilon \cos \beta$$

(see the diagram in Fig. 8). It is easy to see that F_{II} attains its minimum value (whose absolute value is maximum) for $\lambda' \approx -1$, i.e., when the right end of the contact zone is close to the crack tip. In this case, the sole exception is observed for the crack of minimum length ($\varepsilon = 0.062$). The positive maximum values of F_{II} are attained for $\lambda' \approx 1$. In this case, the left end of the contact zone is close to the crack tip. Note that the numerical results obtained by the authors for this case coincide with data obtained by Keer and Bryant.

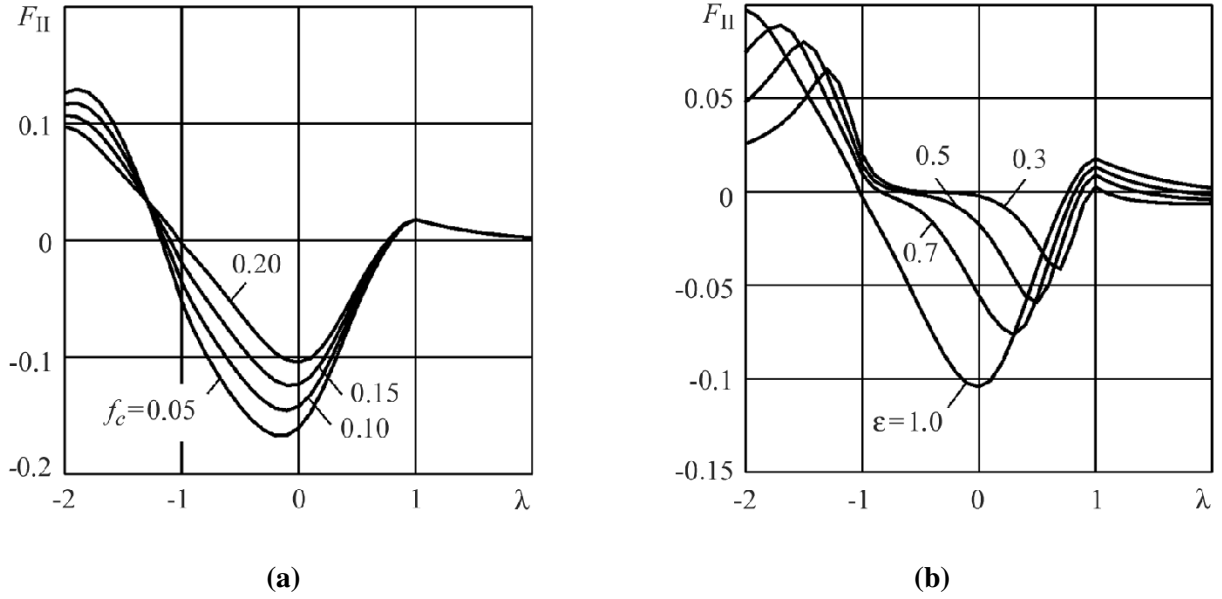


Fig. 9. Dependences of the dimensionless SIF $F_{II} = K_{II}/(p_0\sqrt{\pi a})$ on the location of the load: (a) for various friction coefficients f_c , (b) for various relative crack lengths ε , $\beta = 150^\circ$, $f_s = 0.1$ [25].

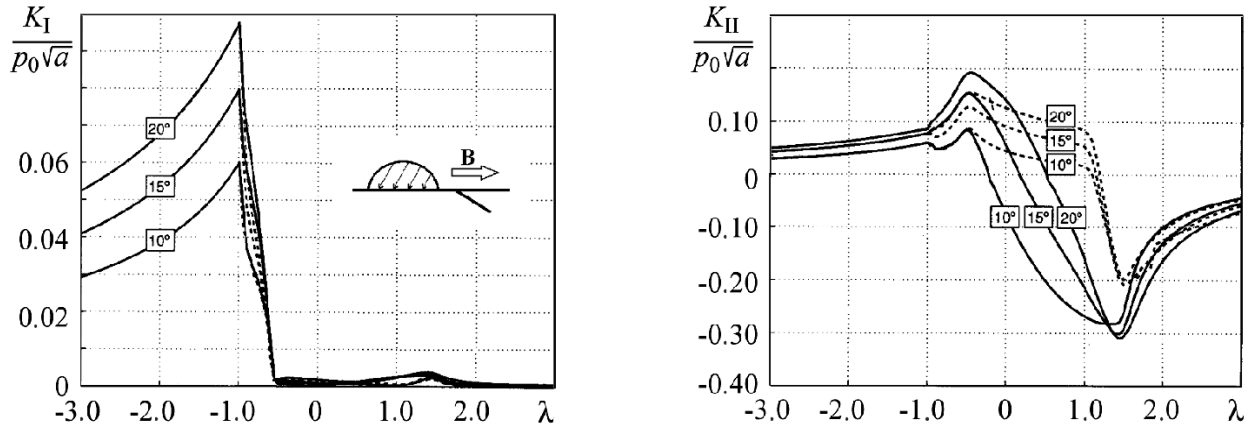


Fig. 10. Dependences of the dimensionless SIF $K_{I,II}/(p_0\sqrt{a})$ obtained by Ringsberg and Bergkvist on the location of the load; $f_s = -0.4$, $\varepsilon = 0.5$ [3]. The continuous lines correspond to $f_c = 0.2$ and the dashed lines correspond to $f_c = 0.5$.

Contact of the Crack Lips with Friction. As indicated above, in the general case, the boundary conditions of the problem of contact of crack lips have the form (2) on the edge of the half plane and the form (3)–(6) on the crack lips.

The influence of friction between the crack lips and crack length on the SIF K_{II} was studied in [25]. As could be expected, the stress intensity factor in shear $|F_{II}|$ is a decreasing function of friction between the crack lips (Fig. 9a). However, for the analyzed values of ε , the indicated parameter increases with the relative crack length (Fig. 9b).

Ringsberg and Bergkvist [3] studied the influence of orientation of the crack on the SIF K_I and K_{II} for two values of the coefficient of friction between the crack lips $f_c = 0.2$ and $f_c = 0.5$ (Fig. 10).

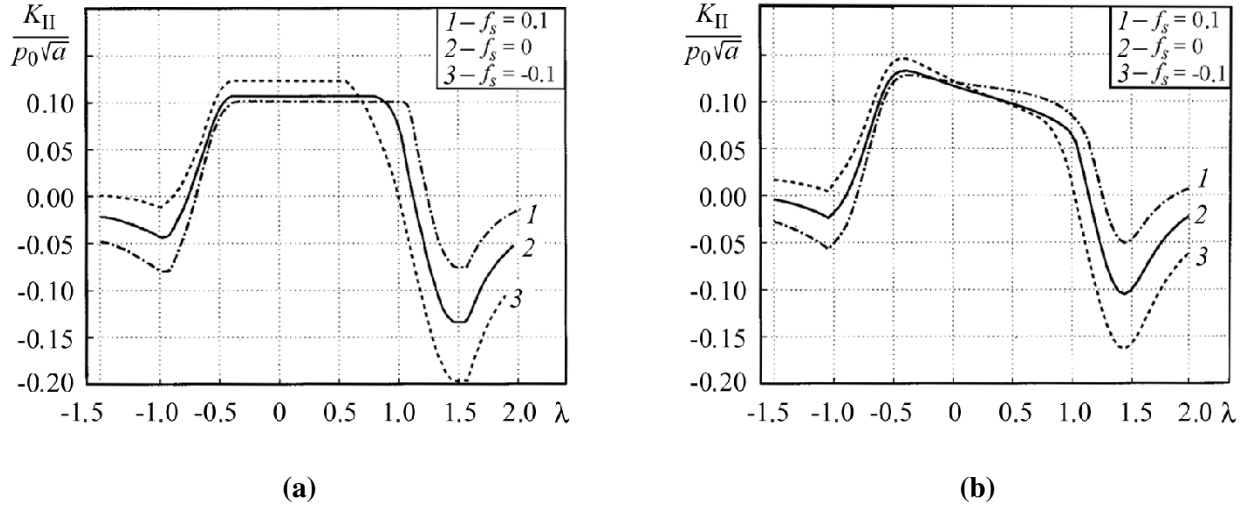


Fig. 11. Comparison of the dimensionless SIF $K_{II}/(p_0\sqrt{a})$ found by Bower [15] (a) and Ringsberg and Bergkvist [3] (b); $f = 0.1$, $\beta = 25^\circ$, and $\varepsilon = 0.5$.

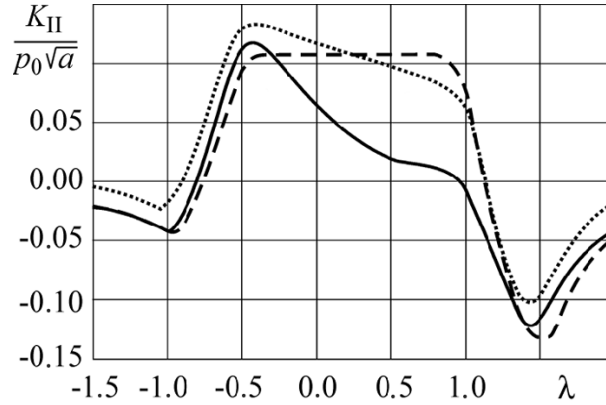


Fig. 12. Comparison of the dimensionless SIF $K_{II}/(p_0\sqrt{a})$ found by Bower (dashed curve) [15] with the results obtained by Ringsberg and Bergkvist (dotted curve) [3] and our data (continuous line); $f_s = 0$, $f_c = 0.1$, $\beta = 25^\circ$, and $\varepsilon = 0.5$.

For all analyzed angles of orientation of the crack, the factors K_I are maximum for $\lambda' = -1$ when the right end of the contact load moving in the left–right direction covers the crack mouth. Moreover, the larger the angle of inclination of the crack to the edge of the plane, the higher the values of these factors. The factors K_{II} take their maximum absolute values when the load completely passes through the crack mouth (for $\lambda \approx 1.5$). Recall that our results are compared with the available literature data [3] for the case where the crack lips are in contact ($\lambda = -1$) in Table 2.

Numerous authors [3, 19, 27, 29] compare their data on the SIF with the well-known results obtained by Bower [15], e.g., in [3] (Fig. 11). It is easy to see that the curves plotted by Ringsberg and Bergkvist for the SIF lie above the corresponding curves plotted by Bower.

We compare our numerical results with the data on the SIF obtained by Bower [15] and Ringsberg and Bergkvist [3] for $f_s = 0$ (Fig. 12). Moreover, in Table 4, for the sake of comparison, we also present the maximum and minimum values of the factor $K_{II}/(p_0\sqrt{a})$ showing that our data are closer to Bower's results than to the data obtained by Ringsberg and Bergkvist.

Table 4. Comparison of the SIF obtained by Bower [15] and Ringsberg and Bergkvist [3] with our data; $f_s = 0$, $f_c = 0.1$, $\beta = 25^\circ$, and $\varepsilon = 0.5$

Reference	$K_{II, \max} / (p_0 \sqrt{a})$	$K_{II, \min} / (p_0 \sqrt{a})$
[15]	0.11 ($-0.39 < \lambda < 0.79$)	-0.13 ($\lambda = 1.5$)
Data of the authors	0.12 ($\lambda = -0.43$)	-0.12 ($\lambda = 1.43$)
[3]	0.13 ($\lambda = -0.41$)	-0.11 ($\lambda = 1.42$)

On the basis of the comparative analysis of the results obtained by different authors by using different methods, we conclude that the numerical values of the SIF are somewhat different but the general trends of their behavior in the process of motion of the contact load are preserved.

CONCLUSIONS

We present a survey of literature on the SIF for a shallow edge crack in the elastic half plane subjected to the action of Hertzian load, which is the most extensively used model problem in the investigation of contact rolling interaction. It is shown that the results obtained by different authors by using different methods are somewhat different but the general trends in the behavior of the SIF in the process motion of the contact load are preserved. It is shown that our data obtained by the method of singular integral equations are in better agreement with the results obtained by Keer with coauthors and Bower by using the dislocation approach and satisfactorily agree with the results obtained by the finite-element method.

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