

STRENGTH CALCULATIONS AND OPTIMAL WEIGHT DESIGN OF MULTILAYER SHELL-SHAPED COMPOSITE PRODUCTS UNDER A SET OF LOADS

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Abstract: The stress–strain state of axisymmetric multilayer shells is analyzed using kinematic and static hypotheses that allow for the transverse shear stresses satisfying the necessary equations of state, continuity conditions at the boundaries between the layers and given boundary conditions. A numerical solution of the problem of the stress–strain state for a multilayer bar is compared with the Lekhnitskii solution (for a cantilever beam loaded by a concentrated force and moment) to assess the applicability of the employed bending equations of multilayer shells. It is shown that these solutions are in good agreement. The problem of the initial fracture of the shells considered is formulated using phenomenological strength criteria for each layer. A coordinate-wise descent method in the unit interval is proposed to solve weight optimization problems for multilayer shells of composite materials under combined loading. Regions of safe operating loads and the optimal weight distribution of layer thicknesses are determined for a multilayer bar acted upon by a uniformly distributed load and concentrated force.

Keywords: multilayer shells, composite material, set of loads, strength, optimal weight design, coordinate-wise descent method in the unit interval.

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INTRODUCTION

The results of the main theoretical studies of layered anisotropic shells and plates are presented in [1–7]. In [6], methods are described that are used to derive two-dimensional equations of layered anisotropic shells: the method of asymptotic integration of the equations of the spatial problem of elasticity, the method of expansion in series in a certain system of functions of the transverse coordinate, the method of hypotheses for each layer or a stack of layers in combination with the Lagrange and Reissner variational principles. In accordance with the kinematic hypothesis proposed in [6], the distribution of shear stresses along the transverse coordinate is such that their maximum value is always reached on the middle surface of the shell regardless of the structure of the multilayer stack. This is contrary to the results obtained for multilayer bars [7]. In [7], the distributions of stresses and displacements in the case of bending of layered beams were obtained from the equilibrium equations for a continuous medium using a stress function in the form of integer polynomials. In this way, it is possible to obtain a solution for the case of an orthotropic cantilever beam bent by a force and momentum.

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1. FORMULATION OF THE BASIC STATIC AND KINEMATIC HYPOTHESES

We consider orthotropic axisymmetric shells whose loading and fastening does not depend on the angular coordinate φ and the axes of orthotropy coincide with the lines of curvature of the reference surface. In this case, all the quantities are independent of φ and the shear strains are zero: $\varepsilon_{x\varphi} = \varepsilon_{z\varphi} = 0$. In this paper, the following notation is used: OX and OZ are the coordinate axes directed along the meridian and the normal to the reference surface of the shell, respectively, u_k and w_k ($k = 1, 2, \dots, n$) are the displacements along the axes OX and OZ of the k th layer, σ_{xk} , $\sigma_{\varphi k}$, and σ_{xzk} are the normal and shear stresses in the k th layer, the reference surface is the upper front surface of the shell, h_k is the distance from the upper edge of the shell to the lower boundary of the k th layer ($h_0 = 0$, $h_n = h$, n is the number of layers), and R_1 , R_2 and A_1 , A_2 are the radii of curvature and Lamé parameters of the reference surface of the shell, respectively.

Results of experiments [8–12] show that in many cases, fracture due to transverse shear stresses limits the bearing capacity of structures of composite materials. Accounting for compression has a significantly less effect on the stress–strain state of the structure than accounting for transverse shears [9]. Therefore, we assume that

$$\varepsilon_{zk} = 0, \quad w_k = w(x), \quad \sigma_{zk} \ll \sigma_{xk}, \quad \sigma_{zk} \ll \sigma_{\varphi k}, \quad \sigma_{zk} \ll \sigma_{xzk}, \quad k = 1, 2, \dots, n. \quad (1)$$

In this case, the generalized Hooke's law equation for the k th layer can be written as

$$\sigma_{xk} = a_{11k}\varepsilon_{xk} + a_{12k}\varepsilon_{\varphi k}, \quad \sigma_{\varphi k} = a_{12k}\varepsilon_{xk} + a_{22k}\varepsilon_{\varphi k}, \quad \sigma_{xzk} = a_{13k}\varepsilon_{xzk}. \quad (2)$$

In the general case, the coefficients $a_{\alpha\alpha k}$ and a_{1ik} ($k = 1, 2, \dots, n$, $\alpha = 1, 2$, and $i = 2, 3$) can be arbitrary continuous functions of the coordinate z .

Summarizing the results [7] for the case of axisymmetric shells, as in [13], the static hypothesis [along with assumptions (1)] is taken to be the following distribution along the coordinate z of transverse shear stresses for the k th layer:

$$\sigma_{xzk} = \gamma(x) \sum_{i=1}^3 i z^{i-1} C_{ik} \quad \forall z \in [h_{k-1}; h_k], \quad k = 1, 2, \dots, n. \quad (3)$$

Here the constants C_{ik} are given by the following formulas [7]:

$$C_{1k} = \frac{6}{S} \left[S_1 \left(\sum_{i=1}^{k-1} (h_i^2 - h_{i-1}^2) a_i - h_{k-1}^2 a_k \right) - S_2 \left(\sum_{i=1}^{k-1} (h_i - h_{i-1}) a_i - h_{k-1} a_k \right) \right],$$

$$C_{2k} = -\frac{3S_2 a_k}{S}, \quad C_{3k} = \frac{2S_1 a_k}{S},$$

$$S_l = \sum_{k=1}^n (h_k^l - h_{k-1}^l), \quad l = 1, 2, 3, \quad S = 4S_1 S_3 - 3S_2^2, \quad a_i = a_{11k} + a_{22k}.$$

Substituting the third relation (2) for the transverse shear strain into (3) and integrating the resulting relation, we have the following representation for the displacement along the coordinate z of the k th layer:

$$u_k(x, z) = u(x) - \frac{z}{A_1} \frac{dw}{dx} + \frac{\gamma(x)}{a_{13k}} \sum_{\alpha=0}^3 z^\alpha C_{\alpha k}, \quad \alpha = 0, 1, 2, 3, \quad k = 1, 2, \dots, n. \quad (4)$$

Here

$$C_{01} = 0, \quad C_{0k+1} = \frac{a_{13k+1}}{a_{13k}} \sum_{\alpha=0}^3 h_k^\alpha C_{\alpha k} - \sum_{\alpha=1}^3 h_k^\alpha C_{\alpha k+1}.$$

In (4) and below, quantities of the order of z/R_i can be ignored due to their smallness compared with unity.

The distribution of the meridional displacement (4) takes into account the transverse shear stresses (in each layer) satisfying zero boundary conditions at $z = 0$ and $z = h$ and provides the satisfaction of all the continuity conditions for meridional displacements and shear stresses at the interfaces between layers at $z = h_k$ ($k = 1, 2, \dots, n-1$).

2. EQUILIBRIUM EQUATIONS AND BOUNDARY CONDITIONS OF THE LINEAR THEORY OF MULTILAYER ANISOTROPIC SHELLS

To derive differential equilibrium equations for the shells considered, we apply the principle of possible displacements, which allows obtaining a system of generalized internal forces corresponding to the independent generalized kinematic characteristics of the adopted model, the equilibrium equations, and the natural boundary conditions consistent with the accepted hypotheses.

For a multilayer anisotropic shell in equilibrium, the principle of virtual displacements has the form

$$\delta\Pi = \delta E + \delta A, \quad (5)$$

where $\delta\Pi$ is the virtual work of internal forces; δE is the virtual work of external distributed loads acting on the faces of the shell $z = 0$ and $z = h$; δA is the virtual work of the forces applied to the ends of the shell $x = l_1$, $x = l_2$.

The variation of the strain potential energy is expressed as

$$\delta\Pi = \int_0^{2\pi} \int_{l_1}^{l_2} \sum_{i=1}^n \int_{h_{i-1}}^{h_i} (\sigma_{xi} \delta\varepsilon_{xi} + \sigma_{\varphi i} \delta\varepsilon_{\varphi i} + \sigma_{xzi} \delta\varepsilon_{xzi}) A_1 A_2 dz dx d\varphi. \quad (6)$$

The virtual works δE and δA are given by

$$\begin{aligned} \delta E &= \int_0^{2\pi} \int_{l_1}^{l_2} q(x) \delta w A_1 A_2 dx d\varphi, \\ \delta A &= \int_0^{2\pi} \sum_{\alpha=1}^2 \sum_{i=1}^n \int_{h_{i-1}}^{h_i} \left(P_{xi}^0(l_\alpha, z) \delta u_i(l_\alpha, z) + P_{zi}^0(l_\alpha, z) \delta w(l_\alpha, z) \right) A_2 dz d\varphi. \end{aligned} \quad (7)$$

Here $q(x)$ is the distributed load applied to the reference surface of the shell; $P_{xi}^0(l_\alpha, z)$ and $P_{zi}^0(l_\alpha, z)$ are the distributed loads in the i th layer ($i = 1, 2, \dots, n$), applied in the sections $x = l_\alpha$ ($\alpha = 1, 2$) and directed along the axes OX and OZ , respectively.

Using (4)–(7) and the Cauchy strain relations and taking into account that all the quantities are independent of the angular coordinate φ , and the variations δu , δw , and $\delta\gamma$ in the integrals over the variable x (after integration by parts) and the variation of δu , δw , $\delta w'$, and $\delta\gamma$ on the boundaries of $x = l_1$ and $x = l_2$ are independent, we obtain a system of differential equations of equilibrium of an element of the multilayer shell

$$\begin{aligned} (A_2 T_x)' - A_2' T_\varphi - \frac{A_1 A_2}{R_1} \tilde{Q} &= 0, \quad \frac{T_x}{R_1} + \frac{T_\varphi}{R_2} + \frac{(A_2 \tilde{Q})'}{A_1 A_2} + q(x) = 0, \\ (A_2 M_x)' - A_2' M_\varphi - A_1 A_2 \tilde{Q} &= 0 \end{aligned} \quad (8)$$

and the boundary conditions at the ends $x = l_1$, $x = l_2$

$$\begin{aligned} [T_x - T_x^0] \delta u &= 0, \quad [(A_2 M_x)' - A_2' M_\varphi - A_1 A_2 \tilde{Q}] \delta w = 0, \\ [M_x - M_x^0] \delta(w') &= 0, \quad [\tilde{M}_x - \tilde{M}_x^0] \delta\gamma = 0. \end{aligned} \quad (9)$$

Here

$$\begin{aligned} T_{x(\varphi)} &= \sum_{k=1}^n \int_{h_{k-1}}^{h_k} \sigma_{x(\varphi)k} dz, \quad M_{x(\varphi)} = \sum_{k=1}^n \int_{h_{k-1}}^{h_k} \sigma_{x(\varphi)k} z dz, \\ \tilde{Q} &= \sum_{k=1}^n \int_{h_{k-1}}^{h_k} \frac{\sigma_{xzk} f_k(z)}{a_{13k}} dz - \frac{1}{A_1 A_2} \{ [A_2 (\tilde{M}_x - M_x)]' - A_2' (\tilde{M}_\varphi - M_\varphi) \}, \\ \tilde{M}_{x(\varphi)} &= \sum_{k=1}^n \int_{h_{k-1}}^{h_k} \sigma_{x(\varphi)k} \Phi_k(z) dz, \quad T_x^0 = \sum_{k=1}^n \int_{h_{k-1}}^{h_k} P_{xk}^0 dz, \quad M_x^0 = \sum_{k=1}^n \int_{h_{k-1}}^{h_k} P_{xk}^0 z dz, \end{aligned}$$

$$Q^0 = \sum_{k=1}^n \int_{h_{k-1}}^{h_k} P_{zk}^0 dz, \quad \tilde{M}_x^0 = \sum_{k=1}^n \int_{h_{k-1}}^{h_k} P_{xk}^0 \Phi_k(z) dz,$$

$$f_k(z) = \sum_{i=1}^3 i z^{i-1} C_{ik}, \quad \Phi_k(z) = \frac{1}{a_{13k}} \sum_{i=0}^3 z^i C_{ik} \quad (k = 1, \dots, n),$$

prime denotes the derivative with respect to x .

From relations (9), we can obtain different variants of the boundary conditions for the class of external loads acting on the shell. In particular, in the case of a fixed-end shell, we have $u = w = w' = \gamma = 0$. These conditions, in contrast to the conditions adopted in the theory discussed in [6], correspond to zero displacement in the meridional direction across the section of the shell. If the edge of the shell is acted upon by distributed forces, the boundary conditions for the forces and moments in (9) are satisfied.

Using relations (4), the strain expression, and the generalized Hooke's law (1), we reduce system (8) to the resolving system of three differential equations of the eighth order (for any number of layers) for the functions $u(x)$, $w(x)$, and $\gamma(x)$.

It should be noted that using the generalized Timoshenko hypotheses, the resulting equations for $n = 1$ are transformed into the equilibrium equations for homogeneous axisymmetric shells [14].

3. INITIAL FRACTURE OF SHELLS UNDER COMBINED LOADING

Once the solution of system (8) with the corresponding boundary conditions has been obtained, we can determine the stress in each shell layer using the generalized Hooke's law relation (1). In solving the problem of the initial fracture of the shell, we use a phenomenological strength criterion that for the k th layer in the general case of a plane stress state has the form [10, 15]

$$f_{11(k)}\sigma_{xk} + f_{22(k)}\sigma_{\varphi k} + f_{1111(k)}\sigma_{xk}^2 + f_{2222(k)}\sigma_{\varphi k}^2 + 2f_{1122(k)}\sigma_{xk}\sigma_{\varphi k} + f_{1212(k)}\sigma_{xz k}^2 = 1, \quad (10)$$

where the coefficients $f_{\alpha\alpha(k)}$, $f_{\alpha\alpha\alpha\alpha(k)}$, $f_{\alpha\beta\alpha\beta(k)}$, and $f_{\alpha\alpha\beta\beta(k)}$ ($\alpha = 1, 2$, $\beta = 1, 2$, $k = 1, 2, \dots, n$) were defined in [15] using experimental data; $\sigma_{1(k)}^\pm$, $\sigma_{2(k)}^\pm$, and $\tau_{12(k)}$ are the strength limits in the reinforcement direction and transverse direction and the shear strength limit in the plane of the layer for the unidirectional reinforced layer (the plus and minus correspond to extension and compression, respectively).

Using the strength criteria (10), we formulate the problem of fracture of the shell considered under combined loading p_j ($j = 1, 2, \dots, N$) as follows: for the given geometric parameters of the shell, the number of its layers, their relative thicknesses, and the mechanical characteristics of each layer, it is required to determine a closed surface (strength surface) in the N -dimensional space of loads p_j ($j = 1, 2, \dots, N$) in such a manner that if the load vector is within this surface, the shell deforms elastically, and if the load vector lies on the surface, fracture begins in at least one of the shell layers. It should be noted that if the load vector lies on the surface, this may correspond to the occurrence of fracture in several layers simultaneously and in different regions (points) of the shell. To construct this strength surface, we use the principle of independence of the forces and the algorithm given in [16].

4. STRESS-STRAIN STATE AND STRENGTH OF A CANTILEVER BEAM

As an example, we consider a cantilever multilayer beam under the action of a system of independent loads: a distributed sinusoidal load p_0 , a uniformly distributed load p_1 , a concentrated force p_2 , and a moment p_3 at the end. For this structure, a method of numerical study of the stress-strain state and initial failure was developed. The solution of the problem of the stress-strain state was compared with the Lekhnitskii solution [7] for a two-layer beam ($n = 2$) under the action of a concentrated force p_2 applied at the end for the following parameter values: $\beta = 10$, $\bar{a}_{111} = 2$, $\bar{a}_{112} = 18$, $\bar{a}_{131} = 1$, and $\bar{a}_{132} = 9$ ($\beta = l/h$ is the relative length of the beam and $\bar{a}_{1\alpha k} = a_{1\alpha k}/a_{131}$, where $\alpha = 1, 3$ and $k = 1, 2$). Figure 1a shows the distribution of the normal stresses $\bar{\sigma}_{x1} = h\sigma_{x1}/p_2$ on the axis of the bar on its surface for $\eta = z/h = 0$, and Fig. 1b shows the distribution of the normal stresses $\bar{\sigma}_{xk} = h\sigma_{xk}/p_2$ over the bar section in the fastening at $\xi = x/l = 0$.

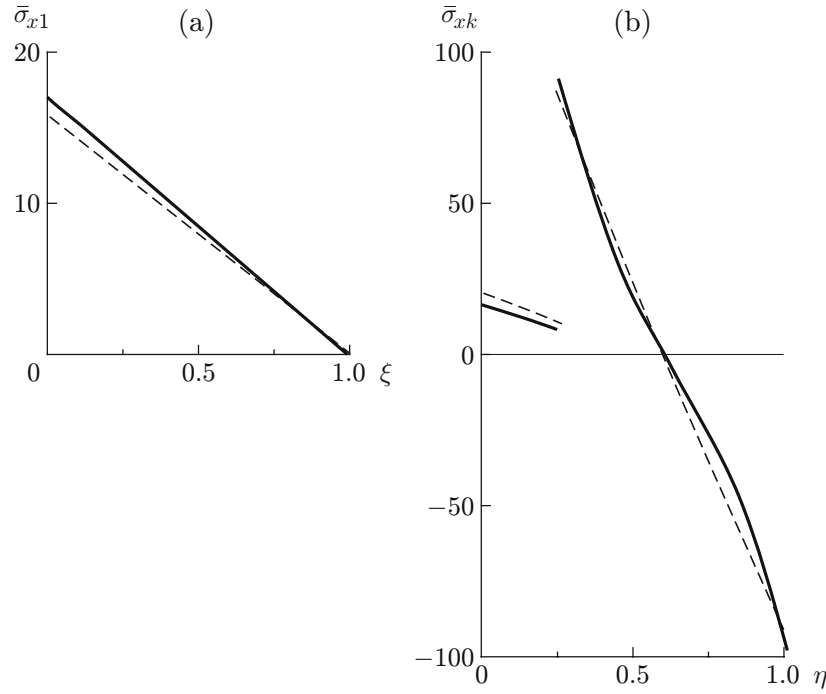


Fig. 1. Distribution of the normal stress along the axis of the bar on the bar surface $\eta = 0$ (a) and over the bar section in the fastening $\xi = 0$ (b): solid curves are the calculation results obtained in this work; dashed curves are the results of calculations by the Lekhnitskii relations [7].

Figure 2 shows the boundary of the region of fracture loads $p_2 = p_2(p_1)$ in the case where the beam is acted upon by a uniformly distributed and concentrated load for $n = 4$, $\bar{a}_{112} = \bar{a}_{113} = 2$, $\bar{a}_{131} = \bar{a}_{134} = 1$, $\bar{a}_{132} = \bar{a}_{133} = 0.5$, $\beta = 10$, $\bar{a}_{111} = \bar{a}_{114} = 10$, $\sigma_{1(1)}^+ = \sigma_{1(4)}^+ = 1800$ MPa, $\sigma_{1(1)}^- = \sigma_{1(4)}^- = 600$ MPa, $\sigma_{1(2)}^+ = \sigma_{1(3)}^+ = 60$ MPa, $\tau_{12(k)} = 45$ MPa, and $\sigma_{1(2)}^- = \sigma_{1(3)}^- = 160$ MPa ($k = 1, 2, 3, 4$).

5. OPTIMAL WEIGHT DESIGN OF MULTI-LAYER SHELL-SHAPED ARTICLES OF COMPOSITE MATERIALS. FORMULATION OF THE PROBLEM, BASIC ASSUMPTIONS

Suppose the following parameters of the structure are known:

- number n layers of given orientation (by the layer we mean the set of all monolayers of the same orientation);
- elastic characteristics of the composite material: elastic moduli E_1 , E_2 , and G_{12} , Poisson's ratios ν_1 and ν_2 , and the specific density of the material of the k th layer γ_k ;
- tensile strengths $\sigma_{1(k)}^\pm$, $\sigma_{2(k)}^\pm$, and $\tau_{12(k)}$ for a unidirectionally reinforced layer;
- constant orientation φ_k ($k = 1, \dots, n$) of each layer relative to the chosen coordinate system XOY ;
- static loads $p^l = (p_0^l, p_1^l, p_2^l, p_3^l)$ ($l = 1, \dots, L$) acting on the structure.

The variable parameters are the thickness of the layers h_k . It is necessary to determine the minimum weight of the composite shell structure

$$V = \sum_{k=1}^n h_k \gamma_k \rightarrow \min, \quad h_k \geq 0$$

provided that the strength conditions is satisfied for all layers and for all L variants of static loads for a given fracture criterion.

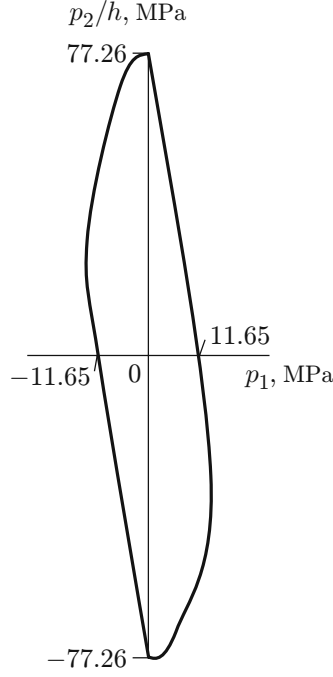


Fig. 2. Boundary of the region of fracture loads for a cantilever beam under the action of a concentrated load and a concentrated force at the end.

To solve the problem of optimizing the structure, it is reasonable to use a phenomenological strength criterion that for the k th layer is written as [15]

$$G_{ji}^k \varepsilon_j^l \varepsilon_i^l + G_j^k \varepsilon_j^l \leq 1, \quad k = 1, \dots, n, \quad l = 1, \dots, L, \quad j = 1, 2, 3, \quad i = 1, 2, 3. \quad (11)$$

Here $\varepsilon_1^l = \varepsilon_x^l$, $\varepsilon_2^l = \varepsilon_y^l$, and $\varepsilon_3^l = \gamma_{xy}^l$ are the strains of the stack of layers in the coordinate system XOY in the l th variant of loading. The constant G_{ji}^k and G_j^k are calculated in terms of the elastic and strength characteristics of the material and the orientation angles of the k th layer.

We introduce the notation

$$\Phi_l^k = G_{ji}^k \varepsilon_j^l \varepsilon_i^l + G_j^k \varepsilon_j^l, \quad \Phi = \max_{k,l} \{\Phi_l^k\}, \quad k = 1, \dots, n, \quad l = 1, \dots, L.$$

Then, the set of constraints of the type of (11) can be written as

$$\Phi(h_1, \dots, h_n) \leq 1. \quad (12)$$

Next, we divide the structure in the meridional direction with a predetermined step and use formula (12) for each node.

6. COORDINATE-WISE DESCENT METHOD IN THE UNIT INTERVAL FOR THE WEIGHT DESIGN OF COMPOSITE SHELLS

To solve the problem of weight optimization of composite shells, it is proposed to use the coordinate-wise descent method in the unit interval [17]. To develop an effective optimization algorithm, we introduce new design parameters whose range of variation is the unit segment $\omega_k = h_k \gamma_k / V$ ($k = 1, \dots, n$). The values of ω_k belong to the interval $[0, 1]$. We fix a randomly selected number j so that the inequalities $1 \leq j \leq n$ are satisfied. Let

$$\omega^* = \max_{k \neq j} (\omega_k), \quad k = 1, \dots, j-1, j+1, \dots, n.$$

We introduce additional parameters $\tilde{\omega}_k = \omega_k/\omega^*$ ($k = 1, \dots, j-1, j+1, \dots, n$). For known values of V , j , ω_j , and $\tilde{\omega}_k$ ($k \neq j$), the thickness of each layer is determined uniquely [17]:

$$h_j = \frac{\omega_j V}{\gamma_j}, \quad h|_{k \neq j} = \frac{\omega_k V}{\gamma_k} = \frac{\omega^* \tilde{\omega}_k V}{\gamma_k} = (1 - \omega_j) \tilde{\omega}_k V / \left(\gamma_k \sum_{k=1, k \neq j}^n \tilde{\omega}_k \right). \quad (13)$$

We substitute (13) into (12). For fixed values of $\tilde{\omega}_k$ for the envelope function of constraints, we obtain an implicit dependence of the weight of the structure on the relative weight j of the layer $\Phi(\omega_j, V) = 1$.

The weight of the structure is minimized sequentially using the method of weight optimization of composite structures in a finite interval [17].

7. EXAMPLE OF OPTIMIZATION

We consider a cantilever beam of length $l = 10$ m loaded at the free end by a concentrated force $L_2 = 10^6$ N. The beam consists of four layers. The first, second, and fourth layers are made of HMS/3002M material which has the following elastic and strength characteristics: $E_1 = 185$ GPa, $E_2 = 6.76$ GPa, $G_{12} = 5.86$ GPa, $\nu_1 = 0.2$, $X_t = 680$ MPa, $X_c = 690$ MPa, $Y_t = 16$ MPa, $Y_c = 186$ MPa, and $S = 72$ MPa. In the first and fourth layers, the fiber is directed along the longitudinal axis of the beam, and in the second layer, it is normal to this axis. The third layer is made of a light filler with a relative specific weight equal to the ratio of the specific weight of the filler to the specific weight of the bearing layers $\bar{\gamma} = 0.1$ and the following elastic characteristics: $E_1 = 0.76$ GPa, $E_2 = 0.76$ GPa, $G_{12} = 0.86$ GPa, $\nu_1 = 0.2$, $X_t = 7$ MPa, $X_c = 7$ MPa, $Y_t = 7$ MPa, $Y_c = 7$ MPa, and $S = 2$ MPa. By optimizing the problem, we obtained the following thicknesses of the layers: $h_1 = 3.32 \cdot 10^{-2}$ m, $h_2 = 1.24 \cdot 10^{-4}$ m, $h_3 = 5.02 \cdot 10^{-2}$ m, and $h_4 = 2.43 \cdot 10^{-2}$ m.

CONCLUSIONS

Based on the theory used to determine the stress-strain state and the region of permissible operating loads for multilayer composite shells, a method of weight optimization of these structures in a finite interval was proposed. This approach allows the use of effective nonlinear programming methods (bisection method and golden section method). In the presence of constraints on the strength of the layers of composite shells, the shell bending equations and the proposed method of weight optimization can be used to design and optimize a wide class of composite products subjected to a set of loads.

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