

TENSION OF A NONTHIN TRANSVERSELY ISOTROPIC PLATE WITH A NONCIRCULAR CYLINDRICAL CAVITY

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The three-dimensional stress state of a transversely isotropic plate with a nearly circular cylindrical cavity is examined. The cavity surface is subject to normal and tangential stresses and the plate is subject at infinity to tensile and shear forces. The problem is solved by expanding unknown functions into Fourier–Legendre series in the thickness coordinate and using the boundary-shape perturbation method. The equations and recurrence formulas needed to solve the problem in an arbitrary approximation are presented

Keywords: transversely isotropic plate, noncircular cylindrical cavity, unknown functions, Fourier–Legendre series, boundary-shape perturbation method

Introduction. A technique to analyze the stress–strain state of a nonthin transversely isotropic plate with a noncircular cylindrical cavity in the case of antisymmetric (about the mid-surface) deformation is outlined in [15]. This technique is based on the expansion of unknown functions into Fourier–Legendre series in the thickness coordinate [1, 14] and use of the boundary-shape perturbation method [4–7]. The stress–strain state of noncircular cylindrical shells is analyzed in [10–12]. In the present paper, we extend the technique to the case of symmetric deformation of a transversely isotropic plate with a noncircular cylindrical cavity.

1. Equilibrium Equations for a Plate. Consider a plate of thickness $2h$ ($h = \text{const}$) occupying a domain $\Omega = S \times [-h, h]$ in three-dimensional space. The plate is referred to a Cartesian coordinate system x_i ($i = 1, 2, 3$). Let $x_1, x_2 \in S$, and $x_3 \in [-h, h]$. Following [2], we will expand the components of the displacement vector $u_j(x_1, x_2, x_3)$ and stress tensor $\sigma_{ij}(x_1, x_2, x_3)$ into a truncated Fourier–Legendre series:

$$\{u_j(x_1, x_2, x_3), \sigma_{ij}(x_1, x_2, x_3)\} = \sum_{k=0}^N \{u_j^{(k)}(x), h^{-1}\sigma_{ij}^{(k)}(x)\} P_k(\zeta), \quad (1.1)$$

where $P_k(\zeta)$ are Legendre polynomials; $x = (x_1, x_2) \in S$, $\zeta = h^{-1}x_3 \in [-1, 1]$, $u_j^{(k)}(x)$, and $\sigma_{ij}^{(k)}(x)$ are the expansion coefficients to be called moments.

If the boundary planes $S^+(x_3 = h)$ and $S^-(x_3 = -h)$ are free from stresses, then the functions $\sigma_{ij}^{(k)}(x)$ satisfy the system of differential equations

$$\partial_\alpha \sigma_{\alpha i}^{(k)} - h^{-1} \underline{\sigma}_{3i}^{(k)} = 0 \quad (i = 1, 2, 3; k = 0, 1, \dots, N), \quad (1.2)$$

where $\partial_\alpha = \partial / \partial x_\alpha$, $\sigma_{3j}^{(k)} = (2k+1)(\sigma_{3j}^{(k-1)} + \sigma_{3j}^{(k-3)} + \dots)$, with $\sigma_{3j}^{(-k)} = 0$ if $k > 0$; summation is over repeated indices; the Roman symbols take the values 1, 2, and 3; and the Greek symbols, the values 1 and 2.

The elastic relations between the moments of the stress and strain components follow from the generalized Hooke's law. In the case of anisotropy, they have the form

$$\sigma_{ij}^{(k)} = h c_{ijlm} \varepsilon_{lm}^{(k)}, \quad (1.3)$$

where c_{ijls} are elastic constants satisfying symmetry conditions; and $\varepsilon_{ij}^{(k)}$ are the strain moments, which are expressed in terms of the displacement moments $u_j^{(k)}$ as

$$\varepsilon_{\alpha i}^{(k)} = \partial_{\alpha} u_i^{(k)} \quad (\alpha = 1, 2), \quad \varepsilon_{3i}^{(k)} = h^{-1} u_i'^{(k)} \quad (i = 1, 2, 3), \quad (1.4)$$

where $u_j'^{(k)} = (2k+1)(u_j^{(k+1)} + u_j^{(k+3)} + \dots)$, and $u_j^{(n)} = 0$ if $n > N$.

If $N = 2n$ ($n = 1, 2, \dots, < \infty$), then Eqs. (1.2) take the following form in the case of symmetric (about the plane S) deformation:

$$\begin{aligned} \partial_{\alpha} \sigma_{\alpha\beta}^{(0)} &= 0 \quad (\beta = 1, 2), \\ \partial_{\alpha} \sigma_{\alpha\beta}^{(2k)} - (4k+1)h^{-1} \sum_{s=1}^k \sigma_{\beta 3}^{(2s-1)} &= 0 \quad (\beta = 1, 2), \\ \partial_{\alpha} \sigma_{\alpha 3}^{(2k-1)} - (4k-1)h^{-1} \sum_{s=0}^{k-1} \sigma_{33}^{(2s)} &= 0 \quad (k = 1, 2, \dots, n), \end{aligned} \quad (1.5)$$

and relations (1.3) in the case of transverse isotropy become

$$\begin{aligned} \sigma_{11}^{(0)} &= h \left[c_{11} e_{11}^{(0)} + c_{12} e_{22}^{(0)} + c_{13} h^{-1} \sum_{s=1}^n u_3^{(2s-1)} \right], \quad \sigma_{12}^{(0)} = 2h c_{66} e_{12}^{(0)}, \\ \sigma_{22}^{(0)} &= h \left[c_{12} e_{11}^{(0)} + c_{11} e_{22}^{(0)} + c_{13} h^{-1} \sum_{s=1}^n u_3^{(2s-1)} \right], \quad \sigma_{33}^{(0)} = h \left[c_{13} e^{(0)} + c_{33} h^{-1} \sum_{s=1}^n u_3^{(2s-1)} \right], \\ \sigma_{11}^{(2k)} &= h \left[c_{11} \partial_1 u_1^{(2k)} + c_{12} \partial_2 u_2^{(2k)} + (4k+1) c_{13} h^{-1} \sum_{s=k+1}^n u_3^{(2s-1)} \right], \\ \sigma_{12}^{(2k)} &= c_{66} h (\partial_1 u_2^{(2k)} + \partial_2 u_1^{(2k)}), \\ \sigma_{22}^{(2k)} &= h \left[c_{12} \partial_1 u_1^{(2k)} + c_{11} \partial_2 u_2^{(2k)} + (4k+1) c_{13} h^{-1} \sum_{s=k+1}^n u_3^{(2s-1)} \right], \\ \sigma_{33}^{(2k)} &= h \left[c_{13} e^{(2k)} + (4k+1) c_{33} h^{-1} \sum_{s=k+1}^n u_3^{(2s-1)} \right]; \\ \sigma_{\alpha 3}^{(2k-1)} &= c_{44} h \left[\partial_{\alpha} u_3^{(2k-1)} + (4k-1) h^{-1} \sum_{s=k}^n u_{\alpha}^{(2s)} \right] \quad (\alpha = 1, 2; k = 1, 2, \dots, n), \end{aligned} \quad (1.6)$$

where

$$2e_{\alpha\beta}^{(0)} = \varepsilon_{\alpha\beta}^{(0)} + \varepsilon_{\beta\alpha}^{(0)} \quad (\alpha, \beta = 1, 2), \quad c_{11} = c_{1111}, \dots, c_{66} = c_{1212},$$

$$e^{(2k)} = \partial_1 u_1^{(2k)} + \partial_2 u_2^{(2k)} \quad (k = 0, 1, \dots, n). \quad (1.8)$$

Equations (1.5)–(1.7) constitute a closed system of equations for the unknown functions $\sigma_{ij}^{(k)}$ and $u_j^{(k)}$.

2. Representation of the General Solution. Let us outline a method of representing the general solution of the system of equations (1.5)–(1.7). Obviously, the first two equations in (1.5) hold if the stress function $F(x_1, x_2)$ is defined by the formulas

$$\sigma_{11}^{(0)} = \partial_2^2 F, \quad \sigma_{12}^{(0)} = -\partial_1 \partial_2 F, \quad \sigma_{22}^{(0)} = \partial_1^2 F. \quad (2.1)$$

Equations (1.6) with (2.1) yield the strain moments $e_{\alpha\beta}^{(0)}$:

$$e_{11}^{(0)} = \frac{1}{4c_* c_{66}^2 h} (-c_{12} \partial_1^2 F + c_{11} \partial_2^2 F) - \frac{c_{13}}{2c_* c_{66} h} \sum_{s=1}^n u_3^{(2s-1)}, \quad e_{12}^{(0)} = -\frac{1}{2c_{66} h} \partial_1 \partial_2 F,$$

$$e_{22}^{(0)} = \frac{1}{4c_* c_{66}^2 h} (c_{11} \partial_1^2 F - c_{12} \partial_2^2 F) - \frac{c_{13}}{2c_* c_{66} h} \sum_{s=1}^n u_3^{(2s-1)}. \quad (2.2)$$

It follows herefrom that

$$e^{(0)} = \frac{1}{2c_* c_{66} h} \Delta F - \frac{c_{13}}{c_* c_{66} h} \sum_{s=1}^n u_3^{(2s-1)}. \quad (2.3)$$

According to (2.3), the function $\sigma_{33}^{(0)}$ takes the form

$$\sigma_{33}^{(0)} = \frac{c_{13}}{2c_* c_{66}} \Delta F + \frac{c_1 c_{11} c_{66}}{c_* c_{66}} \sum_{s=1}^n u_3^{(2s-1)}, \quad (2.4)$$

where Δ is the two-dimensional Laplacian, $c_* = 1 + c_{12} / c_{66}$, $c_1 = c - c_{66} / c_{11}$, and $c = 1 - c_{13}^2 / c_{11} c_{33}$.

Substituting (2.2) into the strain compatibility equation [13], we obtain the following equality for the function $F(x_1, x_2)$:

$$\Delta \Delta F = \frac{2c_{13} c_{66}}{c_{11}} \sum_{s=1}^n \Delta u_3^{(2s-1)}, \quad (2.5)$$

whence

$$\Delta F = \frac{2c_{13} c_{66}}{c_{11}} \sum_{s=1}^n u_3^{(2s-1)} + \frac{4cc_* c_{66} h}{c_1 c_{11}} u, \quad (2.6)$$

where u is an arbitrary harmonic function.

With (2.6), the moment $\sigma_{33}^{(0)}$ becomes

$$\sigma_{33}^{(0)} = \frac{2cc_{13} h}{c_1 c_{11}} u + cc_{33} \sum_{s=1}^n u_3^{(2s-1)}. \quad (2.7)$$

Let us consider the other equations in (1.5). Substituting the moments (1.7) and (2.7) into these equations, we obtain a system of equations:

$$c_{66} \Delta u_\alpha^{(2k)} + (c_{12} + c_{66}) \partial_\alpha e^{(2k)} + (4k+1)h^{-1}$$

$$\times \left(-c_{44} \sum_{s=1}^k \partial_{\alpha} u_3^{(2s-1)} + c_{13} \sum_{s=k+1}^n \partial_{\alpha} u_3^{(2s-1)} - c_{44} h^{-1} \sum_{s=1}^n \beta_{2s}^{(k)} u_{\alpha}^{(2s)} = 0 \right) \quad (\alpha = 1, 2; k = 1, 2, \dots, n), \quad (2.8)$$

$$\begin{aligned} & c_{44} \Delta u_3^{(2k-1)} + (4k-1)h^{-1} \left(-c_{13} \sum_{s=1}^{k-1} e^{(2s)} + c_{44} \sum_{s=k}^n e^{(2s)} - c_{33} h^{-1} \sum_{s=1}^n (\alpha_{2s-1}^{(k)} + c) u_3^{(2s-1)} \right) \\ & = \frac{2(4k-1)c_{13}}{c_1 c_{11} h} u \quad (k = 1, 2, \dots, n), \end{aligned} \quad (2.9)$$

where $\alpha_{2s-1}^{(k)}$ and $\beta_{2s}^{(k)}$ are absolute constants,

$$\alpha_{2s-1}^{(k)} = \begin{cases} (s-1)(2s+1), & 1 \leq s \leq k, \\ (k-1)(2k+1), & k \leq s \leq n, \end{cases} \quad \beta_{2s}^{(k)} = \begin{cases} s(2s+1), & 1 \leq s \leq k, \\ k(2k+1), & k \leq s \leq n. \end{cases}$$

To construct the general solution of the system of equations (2.8), (2.9), we proceed as follows: differentiate the first equation ($\alpha = 1$) in (2.8) with respect to x_1 and the second equation ($\alpha = 2$) with respect to x_2 and sum the resulting expressions. Considering notation (1.8), we obtain

$$c_{11} \Delta e^{(2k)} + (4k+1)h^{-1} \left(-c_{44} \sum_{s=1}^k \Delta u_3^{(2s-1)} + c_{13} \sum_{s=k+1}^n \Delta u_3^{(2s-1)} - c_{44} h^{-1} \sum_{s=1}^n \beta_{2s}^{(k)} e^{(2s)} \right) = 0 \quad (k = 1, 2, \dots, n). \quad (2.10)$$

Equalities (2.9) and (2.10) constitute a system of equations of the $4n$ th order for the unknown functions $u_3^{(2k-1)}$ and $e^{(2k)}$. Setting

$$\begin{aligned} c_{66} u_3^{(1)} &= -\alpha_1 h u + v_1, & c_{66} u_3^{(2k-1)} &= v_{2k-1} \quad (k = 2, \dots, n), \\ c_{66} h e^{(2k)} &= v_{2k} \quad (k = 1, 2, \dots, n), \end{aligned} \quad (2.11)$$

we obtain a homogeneous system of equations for the functions v_k :

$$\sum_{k=0}^{2n} (\alpha_{pk} - \beta_{pk} h^2 \Delta) v_k = 0 \quad (p = 1, 2, \dots, 2n), \quad (2.12)$$

where $\alpha_1 = 2c_{13}c_{66} / c_1 c_{11} c_{33}$ and α_{pk} and β_{pk} are dimensionless constants, which have simple explicit expressions.

Let us consider the characteristic equation

$$\det ||\alpha_{pk} - k\beta_{pk}|| = 0 \quad (2.13)$$

and assume that its roots k_m ($m = 1, 2, \dots, 2n$) are simple and nonzero. Then, according to [13], the solution of the system of equations (2.12) can be represented in the form

$$v_k = \sum_{m=1}^{2n} c_m^{(k)} V_m, \quad (2.14)$$

where V_m are functions satisfying the equalities

$$\Delta V_m - k_m h^{-2} V_m = 0. \quad (2.15)$$

The constants $c_m^{(k)}$ are determined by the algebraic cofactors of the elements of any row in the determinant $|\alpha_{pk} - k_m \beta_{pk}|_{2n \times 2n}$. With (2.14), functions (2.11) become

$$c_{66}u_3^{(1)} = -\mathfrak{x}_1 hu + \sum_{m=1}^{2n} c_m^{(1)} V_m, \quad c_{66}u_3^{(2k-1)} = \sum_{m=1}^{2n} c_m^{(2k-1)} V_m \quad (k=2,3,\dots,n), \quad (2.16)$$

$$c_{66}e^{(2k)} = h^{-1} \sum_{m=1}^{2n} c_m^{(2k)} V_m \quad (k=1,2,\dots,n). \quad (2.17)$$

Note that Eqs. (2.17) can be represented as

$$c_{66} \left(\partial_1 u_1^{(2k)} + \partial_2 u_2^{(2k)} \right) = h \sum_{m=1}^{2n} a_m^{(2k)} \Delta V_m,$$

whence we can find the moments of displacement components $u_\alpha^{(2k)}$ ($\alpha=1,2$):

$$c_{66}u_1^{(2k)} = -h\partial_2 Y_{2k} + h \sum_{m=1}^{2n} a_m^{(2k)} \partial_1 V_m, \quad c_{66}u_2^{(2k)} = h\partial_1 Y_{2k} + h \sum_{m=1}^{2n} a_m^{(2k)} \partial_2 V_m, \quad (2.18)$$

where $a_m^{(2k)} = k_m^{-1} c_m^{(2k)}$ and Y_{2k} are arbitrary sufficiently smooth real functions.

To determine them, we will use the system of equations (2.8). Substituting (2.16)–(2.18) into (2.8), we obtain

$$\begin{aligned} \partial_2 \left[\Delta Y_{2k} - \frac{(4k+1)c_{44}}{c_{66}h^2} \sum_{l=1}^n \beta_{2l}^{(k)} Y_{2l} \right] &= \frac{2(4k+1)c_{13}c_{44}}{c_1c_{11}c_{33}h} \partial_1 u, \\ \partial_1 \left[\Delta Y_{2k} - \frac{(4k+1)c_{44}}{c_{66}h^2} \sum_{l=1}^n \beta_{2l}^{(k)} Y_{2l} \right] &= -\frac{2(4k+1)c_{13}c_{44}}{c_1c_{11}c_{33}h} \partial_2 u. \end{aligned} \quad (2.19)$$

Let us introduce a conjugate harmonic function v related to the function u by the Cauchy–Riemann equations [8]:

$$\partial_1 u = \partial_2 v, \quad \partial_1 v = -\partial_2 u. \quad (2.20)$$

Integrating Eqs. (2.19) over x_1 and x_2 , we obtain the following system of equations (up to an arbitrary constant):

$$\Delta Y_{2k} - \frac{(4k+1)c_{44}}{c_{66}h^2} \sum_{l=1}^n \beta_{2l}^{(k)} Y_{2l} = \frac{2(4k+1)c_{13}c_{44}}{c_1c_{11}c_{33}h} v \quad (k=1,2,\dots,n). \quad (2.21)$$

If

$$Y_2 = -\mathfrak{x}_2 hv + y_1, \quad Y_{2k} = y_k \quad (k=2,3,\dots,n), \quad (2.22)$$

then we have a homogeneous system of equations for y_k :

$$\sum_{l=1}^n (q_{kl} - \delta_{kl} h^2 \Delta) y_k = 0 \quad (k=1,2,\dots,n), \quad (2.23)$$

where δ_{kl} is the Kronecker delta; $\mathfrak{x}_2 = 2c_{13}c_{66} / 3c_1c_{11}c_{33}$, and $q_{kl} = (4k+1)\beta_{2l}^{(k)} c_{44} / c_{66}$.

As above, we determine the functions y_k from (2.23) and substitute them into (2.22) to obtain the expressions

$$Y_2 = -\mathfrak{x}_2 hv + \sum_{s=1}^n b_s^{(2)} w_s, \quad Y_{2k} = \sum_{s=1}^n b_s^{(2k)} w_s \quad (k=2,\dots,n), \quad (2.24)$$

where w_s are functions satisfying the equalities

$$\Delta w_s - \lambda_s h^{-2} w_s = 0, \quad (2.25)$$

in which λ_s ($k = 1, 2, \dots, n$) are the roots of the characteristic equation

$$\det ||q_{kl} - \lambda \delta_{kl}|| = 0. \quad (2.26)$$

The constants $b_s^{(2k)}$ are determined from the cofactors of the elements of some row of the determinant $|q_{kl} - \lambda_s \delta_{kl}|_{n \times n}$. With (2.24) and (2.20), functions (2.18) become

$$\begin{aligned} c_{66} u_1^{(2k)} &= \delta_*^{(2k)} h^2 \partial_1 u + h \sum_{m=1}^{2n} a_m^{(2k)} \partial_1 V_m - h \sum_{s=1}^n b_s^{(2k)} \partial_2 w_s, \\ c_{66} u_2^{(2k)} &= \delta_*^{(2k)} h^2 \partial_2 u + h \sum_{m=1}^{2n} a_m^{(2k)} \partial_2 V_m + h \sum_{s=1}^n b_s^{(2k)} \partial_1 w_s, \end{aligned} \quad (2.27)$$

where $\delta_*^{(2)} = \alpha_2$ and $\delta_*^{(2k)} = 0$ ($k > 1$).

Substituting moments (2.16) into Eq. (2.6), we obtain

$$\Delta F = 4hu + 2 \sum_{m=1}^{2n} d_m^{(0)} V_m, \quad (2.28)$$

whence

$$F = hU - h^2 \sum_{m=1}^{2n} a_m^{(0)} V_m, \quad (2.29)$$

where U is a biharmonic function, $4u = \Delta U$, $a_m^{(0)} = -2k_m^{-1} d_m^{(0)}$, and $c_{11} d_m^{(0)} = c_{13} \sum_{s=1}^n c_m^{(2s-1)}$.

Let us represent the function U in the form

$$U = z\varphi(z) + z\overline{\varphi(z)} + \psi_*(z) + \overline{\psi_*(z)}, \quad (2.30)$$

where $\varphi(z)$ and $\psi_*(z)$ are arbitrary analytic functions of a complex variable $z = x_1 + ix_2$. Then $u = \varphi'(z) + \overline{\varphi'(z)}$ and $v = -i(\varphi'(z) - \overline{\varphi'(z)})$, and solution (2.27) can be written in complex form:

$$c_{66} u_+^{(2k)} = 2\delta_*^{(2k)} h^2 \overline{\varphi''(z)} + 2h \sum_{m=1}^{2n} a_m^{(2k)} \partial_{\bar{z}} V_m + 2ih \sum_{s=1}^n b_s^{(2k)} \partial_{\bar{z}} w_s, \quad (2.31)$$

where $u_+^{(2k)} = u_1^{(2k)} + iu_2^{(2k)}$ and $2\partial_{\bar{z}} = \partial / \partial x_1 + i\partial / \partial x_2$, the prime denotes differentiation with respect to z , and the overbar denotes a complex-conjugate variable.

The stress moments are defined by

$$\begin{aligned} \sigma_{11}^{(0)} + \sigma_{22}^{(0)} &= \Delta F, \quad \sigma_{11}^{(0)} - \sigma_{22}^{(0)} + 2i\sigma_{12}^{(0)} = -4\partial_{\bar{z}}^2 F, \\ \sigma_{11}^{(2k)} + \sigma_{22}^{(2k)} &= 2h \left[(c_{12} + c_{66}) e^{(2k)} + (4k+1)c_{13} h^{-1} \sum_{s=k+1}^n u_3^{(2s)} \right], \\ \sigma_{11}^{(2k)} - \sigma_{22}^{(2k)} + 2i\sigma_{12}^{(2k)} &= 4h\partial_{\bar{z}}^2 u_+^{(2k)}, \end{aligned}$$

$$\sigma_{13}^{(2k-1)} + i\sigma_{23}^{(2k-1)} = hc_{44} \left[2\partial_{\bar{z}} u_3^{(2k-1)} + (4k-1)h^{-1} \sum_{s=k}^n u_+^{(2s)} \right]. \quad (2.32)$$

The expressions for the moments $\sigma_{33}^{(0)}$ and $\sigma_{33}^{(2k)}$ ($k > 1$) remain the same.

In the domain S , let us introduce polar coordinates r, θ , in addition to the Cartesian coordinates, and use formulas to transform the stress components from one coordinate system to another. Considering functions (2.16), (2.27), and (2.31), we obtain

$$\begin{aligned} \sigma_{rr}^{(0)} + \sigma_{\theta\theta}^{(0)} &= 4h[\varphi'(z) + \overline{\varphi'(z)}] + 2 \sum_{m=1}^{2n} d_m^{(0)} V_m, \quad \sigma_{33}^{(0)} = \sum_{m=1}^{2n} \tilde{d}_m^{(0)} V_m, \\ \sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)} + 2i\sigma_{r\theta}^{(0)} &= -4h \left[z\overline{\varphi''(z)} + \overline{\psi'(z)} - h \sum_{m=1}^{2n} a_m^{(0)} \partial_{\bar{z}}^2 V_m \right] e^{-2i\theta}, \end{aligned} \quad (2.33)$$

$$\begin{aligned} \sigma_{rr}^{(2k)} + \sigma_{\theta\theta}^{(2k)} &= 2 \sum_{m=1}^{2n} d_m^{(2k)} V_m, \quad \sigma_{33}^{(2k)} = 2 \sum_{m=1}^{2n} \tilde{d}_m^{(2k)} V_m, \\ \sigma_{rr}^{(2k)} - \sigma_{\theta\theta}^{(2k)} + 2i\sigma_{r\theta}^{(2k)} &= 4h \left[2\delta_*^{(2k)} h^2 \overline{\varphi''(z)} + 2h \sum_{m=0}^{2n} a_m^{(2k)} \partial_{\bar{z}}^2 V_m + 2ih \sum_{s=1}^n b_s^{(2k)} \partial_{\bar{z}}^2 w_s \right] e^{-2i\theta}, \\ \sigma_{r3}^{(2k-1)} + i\sigma_{\theta 3}^{(2k-1)} &= 2h \left[\sum_{m=1}^{2n} p_m^{(2k-1)} \partial_{\bar{z}} V_m + i \sum_{m=1}^n q_s^{(2k-1)} \partial_{\bar{z}} w_s \right] e^{-i\theta} \quad (k=1, 2, \dots, n), \end{aligned} \quad (2.34)$$

where $\psi(z) = \psi'_*(z)$.

We use formulas (2.32)–(2.34) to write boundary conditions on the boundary L_0 of radius R of a circular hole. If the components of the stress vector $\hat{\sigma}_{rj}(\theta, \xi)$ ($j = r, \theta, 3$) are prescribed on the cylindrical surface $R \times [-h, h]$, then expanding these functions into a Fourier–Legendre series in the thickness coordinate ξ and equating the expansion coefficients $\hat{\sigma}_{rj}^{(m)}(\theta)$ on the boundary $r = R$ to the corresponding moments $\sigma_{rj}^{(m)}(r, \theta)$, we obtain

$$\begin{aligned} \left| \sigma_{rr}^{(2k)}(r, \theta) - i\sigma_{r\theta}^{(2k)}(r, \theta) \right|_{r=R} &= \hat{\sigma}_{rr}^{(2k)}(\theta) - i\hat{\sigma}_{r\theta}^{(2k)}(\theta) \quad (k=0, 1, \dots, n), \\ \sigma_{r3}^{(2k-1)}(r, \theta) \Big|_{r=R} &= \hat{\sigma}_{r3}^{(2k-1)}(\theta) \quad (k=1, 2, \dots, n), \end{aligned} \quad (2.35)$$

where $\hat{\sigma}_{rj}^{(m)}(\theta)$ are the stress moments defined by

$$\hat{\sigma}_{rj}^{(m)}(\theta) = \left(m + \frac{1}{2} \right) \int_{-h}^h \hat{\sigma}_{rj}^{(m)}(\theta, \xi) P_m(\xi) dx_3. \quad (2.36)$$

3. Analytic Method to Solve the Stress Problem for a Plate with a Noncircular Cylindrical Cavity. We will use the above solution to determine the stress state of a nonthin transversely isotropic plate with a noncircular cylindrical cavity whose surface is a nearly circular cylindrical one of radius R . When projected onto the plane S , it is an infinite domain whose boundary L differs a little from the circle L_0 . To solve the problem, we will use boundary-shape perturbation, which is an efficient approximate method first proposed in [4] to determine the stress concentration around curvilinear holes in thin shells. Later, this method was extended to three-dimensional problems of elasticity for orthogonal noncanonical domains [6, 7].

Let the smooth curvilinear boundary L of the domain S be described by the equations

$$x = \rho \cos \vartheta + \varepsilon \rho^{-l} \cos l\vartheta, \quad y = \rho \sin \vartheta - \varepsilon \rho^{-l} \sin l\vartheta, \quad (3.1)$$

where $x = x_1 / R$ and $y = x_2 / R$ are dimensionless Cartesian coordinates; ρ and ϑ are orthogonal curvilinear coordinates; and ε and l are parameters describing the shape of the hole. Equations (3.1) have the following complex form:

$$x + iy = \omega(\zeta) = \zeta + \varepsilon f(\zeta) = r e^{i\beta}, \quad (3.2)$$

where $f(\zeta) = \rho^{-l} e^{-il\vartheta}$, $\zeta = \rho e^{i\vartheta}$.

Since the coordinate system (ρ, θ, ξ) is rotated through an angle β relative to (r, θ, x_3) about the common axis $\xi = x_3 / h$, and expansions of the form (1.1) hold in both systems, the well-known transformation formulas for the components of the stress tensor in the domain S yield

$$\begin{aligned} \sigma_{\rho\rho}^{(2k)} - \sigma_{\vartheta\vartheta}^{(2k)} - 2i\sigma_{\rho\vartheta}^{(2k)} &= \left(\sigma_{rr}^{(2k)} - \sigma_{\theta\theta}^{(2k)} - 2i\sigma_{r\theta}^{(2k)} \right) e^{2i\beta}, \\ \sigma_{\rho\rho}^{(2k)} + \sigma_{\vartheta\vartheta}^{(2k)} &= \sigma_{rr}^{(2k)} + \sigma_{\theta\theta}^{(2k)} \quad (k = 0, 1, \dots, n), \\ \sigma_{\rho\xi}^{(2k-1)} - i\sigma_{\vartheta\xi}^{(2k-1)} &= \left(\sigma_{r3}^{(2k-1)} - i\sigma_{\theta3}^{(2k-1)} \right) e^{i\beta} \quad (k = 1, 2, \dots, n), \end{aligned} \quad (3.3)$$

where $\sigma_{ij'}^{(m)} = \sigma_{ij'}^{(m)}(\rho, \vartheta)$ ($i, j' = \rho, \vartheta, \xi$) and $\sigma_{ij}^{(m)} = \sigma_{ij}^{(m)}(r, \theta)$ ($i, j = r, \theta, 3$).

According to Eqs. (3.2), the variables r, θ and ρ, ϑ are related by

$$r = |\omega(\zeta)|, \quad \theta = \arctan \frac{\operatorname{Im} \omega(\zeta)}{\operatorname{Re} \omega(\zeta)}, \quad e^{i\beta} = \frac{\zeta}{|\zeta|} \cdot \frac{\omega'(\zeta)}{|\omega'(\zeta)|} \cdot \frac{\overline{\omega(\zeta)}}{|\omega(\zeta)|}. \quad (3.4)$$

The boundary conditions on the boundary L of the domain S have the same form as those for a circular plate. For example, given the stress components $\hat{\sigma}_{\rho j'}(\vartheta, \xi)$ on the cylindrical surface $L \times [-h, h]$, we expand them into Legendre series to obtain the following conditions for the expansion coefficients $\hat{\sigma}_{\rho j'}^{(m)}(\vartheta)$ on the boundary L :

$$\sigma_{\rho v'}^{(2k)}(\rho, \vartheta) \Big|_{\rho=1} = \hat{\sigma}_{\rho v'}^{(2k)}(\vartheta) \quad (v' = \rho, \vartheta), \quad \sigma_{\rho\xi}^{(2k-1)}(\rho, \vartheta) \Big|_{\rho=1} = \hat{\sigma}_{\rho\xi}^{(2k-1)}(\vartheta). \quad (3.5)$$

When expressed in terms of the variables ρ and ϑ using (3.4), Eqs. (2.15) and (2.25) are too complicated to find their solution with separable variables. Therefore, following [4, 6], we will seek the solution of the problem in the form of a series in positive powers of the small parameter ε :

$$\sigma_{ij'}^{(m)}(\rho, \vartheta) = \sum_{\tau=0}^{\infty} \varepsilon^{\tau} \sigma_{ij'}^{(m, \tau)}(\rho, \vartheta), \quad u_{j'}^{(m)}(\rho, \vartheta) = \sum_{\tau=0}^{\infty} \varepsilon^{\tau} u_{j'}^{(m, \tau)}(\rho, \vartheta). \quad (3.6)$$

According to (3.4), the exponents $e^{i\beta}$ and $e^{2i\beta}$ are defined by the formulas

$$\begin{aligned} e^{i\beta} &= 1 + \varepsilon q_1(\zeta, \bar{\zeta}) + \frac{\varepsilon^2}{2} [q_1^2(\zeta, \bar{\zeta}) + 2q_2(\zeta, \bar{\zeta})] + \dots, \\ e^{2i\beta} &= 1 + 2\varepsilon q_1(\zeta, \bar{\zeta}) + 2\varepsilon^2 [q_1^2(\zeta, \bar{\zeta}) + q_2(\zeta, \bar{\zeta})] + \dots, \end{aligned} \quad (3.7)$$

and an arbitrary scalar function $\phi^m(r, \theta)$ (in particular, the stress components $\sigma_{ij}^{(m)}(r, \theta)$ ($i, j = r, \theta, 3$)) is represented as

$$\phi^m(r, \theta) = \sum_{p=0}^{\infty} \varepsilon^p \frac{1}{p!} L_p \phi^m(\rho, \vartheta), \quad (3.8)$$

where L_p are operators,

$$L_0 = 1, \quad L_1 = \frac{\bar{\zeta}f(\zeta) + \zeta\overline{f(\zeta)}}{2\rho} \frac{\partial}{\partial\rho} + \frac{\bar{\zeta}f(\zeta) - \zeta\overline{f(\zeta)}}{2\rho} \frac{i}{\rho} \frac{\partial}{\partial\vartheta},$$

$$L_2 = \frac{(\bar{\zeta}f(\zeta) + \zeta\overline{f(\zeta)})^2}{4\rho^2} \frac{\partial^2}{\partial\rho^2} - \frac{(\bar{\zeta}f(\zeta) - \zeta\overline{f(\zeta)})^2}{4\rho^2} \left(\frac{1}{\rho} \frac{\partial}{\partial\rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial\vartheta^2} \right) - \frac{\bar{\zeta}^2 f^2(\zeta) - \zeta^2 \overline{f^2(\zeta)}}{2\rho^2} \frac{\partial}{\partial\rho} \left(\frac{i}{\rho} \frac{\partial}{\partial\vartheta} \right), \quad (3.9)$$

and the functions $q_1(\zeta, \bar{\zeta})$ and $q_2(\zeta, \bar{\zeta})$ are given by

$$q_1(\zeta, \bar{\zeta}) = \frac{i}{\rho^2} \text{Im}[\zeta\overline{f(\zeta)} + \zeta\bar{\zeta}f'(\zeta)], \quad q_2(\zeta, \bar{\zeta}) = \frac{i}{2\rho^4} \text{Im}[\bar{\zeta}^2 f^2(\zeta) - \zeta^2 \bar{\zeta}^2 \overline{f'^2(\zeta)}].$$

If

$$\partial_\zeta = \frac{1}{2} e^{-i\vartheta} \left(\frac{\partial}{\partial\rho} - \frac{i}{\rho} \frac{\partial}{\partial\vartheta} \right), \quad \partial_{\bar{\zeta}} = \frac{1}{2} e^{i\vartheta} \left(\frac{\partial}{\partial\rho} + \frac{i}{\rho} \frac{\partial}{\partial\vartheta} \right),$$

then the operators L_p become

$$L_0 = 1, \quad L_1(\partial_\zeta, \partial_{\bar{\zeta}}) = f(\zeta)\partial_\zeta + \overline{f(\zeta)}\partial_{\bar{\zeta}},$$

$$L_2(\partial_\zeta, \partial_{\bar{\zeta}}) = f^2(\zeta)\partial_\zeta^2 + 2f(\zeta)\overline{f(\zeta)}\partial_\zeta\partial_{\bar{\zeta}} + \overline{f^2(\zeta)}\partial_{\bar{\zeta}}^2. \quad (3.10)$$

Note that the terms of series (3.8) are determined by applying the operators L_n to the functions $\phi^m(\rho, \vartheta)$ that follow from $\phi^m(r, \theta)$ after formal replacement of r, θ by ρ, ϑ .

Considering the expansions of functions (3.6), (3.8) and using Eqs. (3.3) with (3.7), we obtain (after comparing the coefficients of like powers of ϵ) the following relations:

$$\sigma_{\rho\rho}^{(2k, \tau)} + \sigma_{\vartheta\vartheta}^{(2k, \tau)} = \sum_{l=0}^{\tau} \Lambda_1^{(\tau-l)} (\sigma_{rr}^{(2k, l)} + \sigma_{\theta\theta}^{(2k, l)}), \quad \sigma_{\zeta\bar{\zeta}}^{(2k, \tau)} = \sum_{l=0}^{\tau} \Lambda_1^{(\tau-l)} \sigma_{33}^{(2k, l)},$$

$$\sigma_{\rho\rho}^{(2k, \tau)} - \sigma_{\vartheta\vartheta}^{(2k, \tau)} - 2i\sigma_{\rho\vartheta}^{(2k, \tau)} = \sum_{l=0}^{\tau} \Lambda_2^{(\tau-l)} (\sigma_{rr}^{(2k, l)} - \sigma_{\theta\theta}^{(2k, l)} - 2i\sigma_{r\theta}^{(2k, l)}),$$

$$\sigma_{\rho\zeta}^{(2k-1, \tau)} - i\sigma_{\vartheta\bar{\zeta}}^{(2k-1, \tau)} = \sum_{l=0}^{\tau} \Lambda_3^{(\tau-l)} (\sigma_{r3}^{(2k-1, l)} - i\sigma_{\theta\bar{3}}^{(2k-1, l)}), \quad (3.11)$$

where

$$\Lambda_1^{(0)} = \Lambda_2^{(0)} = \Lambda_3^{(0)} = 1, \quad \Lambda_1^{(1)} = L_1(\partial_\zeta, \partial_{\bar{\zeta}}), \quad 2\Lambda_1^{(2)} = L_2(\partial_\zeta, \partial_{\bar{\zeta}}),$$

$$\Lambda_2^{(1)} = 2q_1(\zeta, \bar{\zeta}) + L_1(\partial_\zeta, \partial_{\bar{\zeta}}),$$

$$\Lambda_2^{(2)} = 2q_1(\zeta, \bar{\zeta})[q_1(\zeta, \bar{\zeta}) + L_1(\partial_\zeta, \partial_{\bar{\zeta}})] + 2q_2(\zeta, \bar{\zeta}) + 0.5L_2(\partial_\zeta, \partial_{\bar{\zeta}}),$$

$$\Lambda_3^{(1)} = q_1(\zeta, \bar{\zeta}) + L_1(\partial_\zeta, \partial_{\bar{\zeta}}),$$

$$\Lambda_3^{(2)} = 0.5[q_1^2(\zeta, \bar{\zeta}) + L_2(\partial_\zeta, \partial_{\bar{\zeta}})] + q_2(\zeta, \bar{\zeta}) + q_1(\zeta, \bar{\zeta})L_1(\partial_\zeta, \partial_{\bar{\zeta}}).$$

According to (3.8), the moments appearing on the right-hand sides of Eqs. (3.11) follow from (2.33) and (2.34) after formal replacement of r, θ by ρ, ϑ . If

$$\begin{aligned}\Phi(\zeta) &= \phi'(\zeta), \quad \Psi(\zeta) = \psi'(\zeta), \quad \tilde{\delta}_*^{(2k)} = 2\delta_*^{(2k)} h^2 / R^2, \quad a_m^{*(0)} = a_m^{(0)} h^2 / R^2, \\ a_m^{*(2k)} &= a_m^{(2k)} h^2 / R^2, \quad b_s^{*(2k)} = 2b_s^{(2k)} h^2 / R^2, \quad p_m^{*(2k-1)} = p_m^{(2k-1)} h / R, \quad q_s^{*(2k-1)} = q_s^{(2k-1)} h / R,\end{aligned}$$

then

$$\begin{aligned}\sigma_{rr}^{(0,l)} + \sigma_{\theta\theta}^{(0,l)} &= 4h[\Phi_l(\zeta) + \bar{\Phi}_l(\zeta)] + 2 \sum_{m=1}^{2n} d_m^{(0)} V_m^{(l)}, \quad \sigma_{33}^{(0,l)} = \sum_{m=1}^{2n} \tilde{d}_m^{(0)} V_m^{(l)}, \\ \sigma_{rr}^{(0,l)} - \sigma_{\theta\theta}^{(0,l)} - 2i\sigma_{r\theta}^{(0,l)} &= 4e^{2i\vartheta} \left\{ -h[\bar{\zeta}_5 \Phi'_l(\zeta) + \Psi(\zeta)] + \sum_{m=1}^{2n} a_m^{*(0)} \partial_\zeta^2 V_m^{(l)} \right\}, \\ \sigma_{rr}^{(2k,l)} + \sigma_{\theta\theta}^{(2k,l)} &= 2 \sum_{m=1}^{2n} d_m^{(2k)} V_m^{(l)}, \quad \sigma_{33}^{(2k,l)} = \sum_{m=1}^{2n} \tilde{d}_m^{(2k)} V_m^{(l)}, \\ \sigma_{rr}^{(2k,l)} - \sigma_{\theta\theta}^{(2k,l)} - 2i\sigma_{r\theta}^{(2k,l)} &= 4e^{2i\vartheta} \left\{ h\delta_*^{2k} \Phi_l(\zeta) + \sum_{m=1}^{2n} a_m^{*(2k)} \partial_\zeta^2 V_m^{(l)} - i \sum_{s=1}^n b_s^{*(2k)} \partial_\zeta^2 w_s^{(l)} \right\}, \\ \sigma_{r3}^{(2k-1,l)} - i\sigma_{3\theta}^{(2k-1,l)} &= 2e^{i\vartheta} \left\{ \sum_{m=1}^{2n} p_m^{*(2k-1)} \partial_\zeta V_m^{(l)} - i \sum_{s=1}^n q_s^{*(2k-1)} \partial_\zeta w_s^{(l)} \right\},\end{aligned}\tag{3.12}$$

where the functions $V_m^{(l)}$ and $w_s^{(l)}$ are defined by

$$\Delta V_m^{(l)} - l_m^2 V_m^{(l)} = 0, \quad \Delta w_s^{(l)} - n_s^2 w_s^{(l)} = 0,\tag{3.13}$$

where $l_m^2 = k_m R^2 / h^2$, $n_s^2 = \lambda_s R^2 / h^2$, and Δ is the Laplacian.

The analytic functions $\Phi_l(\zeta)$ and $\Psi_l(\zeta)$ for the external boundary-value problem are assumed to have the form

$$\Phi_l(\zeta) = \sum_{n=1}^{\infty} a_n^{(l)} \zeta^{-n}, \quad \Psi_l(\zeta) = \sum_{n=1}^{\infty} b_n^{(l)} \zeta^{-n}.\tag{3.14}$$

The form of the functions $V_m^{(l)}$ and $w_s^{(l)}$ depends on the roots of the characteristic equations (2.13) and (2.26). For example, for a magnesium material with the following elastic constants [9]: $c_{11} = 0.597$, $c_{12} = 0.262$, $c_{13} = 0.217$, $c_{33} = 0.617$, $c_{44} = 0.164$ (GPa), Eq. (2.13) has two complex-conjugate roots when $N=2$ ($n=1$) and two real positive and four complex-conjugate roots when $N=6$ ($n=3$). Equation (2.26) has one and three real positive roots, respectively.

Denote the real positive root by k_1 , and the complex-conjugate roots by k_2 and k_3 . Then

$$V_1^{(l)} = \sum_{n=-\infty}^{\infty} B_n^{(l)} K_n(l_1 \rho) e^{in\vartheta}, \quad V_2^{(l)} = \sum_{n=-\infty}^{\infty} C_n^{(l)} H_n^{(1)}(l_2 \rho) e^{in\vartheta}, \quad V_3^{(l)} = \sum_{n=-\infty}^{\infty} D_n^{(l)} H_n^{(2)}(l_3 \rho) e^{in\vartheta},\tag{3.15}$$

where $B_n^{(l)}$, $C_n^{(l)}$, and $D_n^{(l)}$ are constants satisfying the equalities $B_{-n}^{(l)} = \bar{B}_n^{(l)}$, $C_{-n}^{(l)} = (-1)^n \bar{D}_n^{(l)}$, and $D_{-n}^{(l)} = (-1)^n \bar{C}_n^{(l)}$; $K_n(l_1 \rho)$, $H_n^{(1)}(l_2 \rho)$, and $H_n^{(2)}(l_3 \rho)$ are the McDonald and Hankel functions of the first and second kinds; and $l_1 = \sqrt{k_1} R / h$, $l_2 = \sqrt{-k_2} R / h$, and $l_3 = \bar{l}_2$.

The functions $w_s^{(l)}$ have similar representations:

$$w_s^{(l)} = \sum_{n=-\infty}^{\infty} \tilde{B}_n^{(s,l)} K_n(n_s \rho) e^{in\vartheta} \quad (s=1,2,3).\tag{3.16}$$

TABLE 1

$\sigma_{\vartheta\vartheta}$											
h/R	ς										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1	3.10	3.10	3.09	3.09	3.08	3.06	3.04	3.00	2.96	2.88	2.76
	3.08	3.08	3.08	3.08	3.06	3.05	3.04	3.00	2.96	2.86	2.76
4	3.05	3.05	3.04	3.03	3.03	3.04	3.05	3.06	3.06	2.95	2.68
	3.04	3.04	3.04	3.04	3.04	3.05	3.05	3.05	3.04	2.97	2.63

The constants $a_n^{(l)}$, $b_n^{(l)}$, $B_n^{(l)}$, $C_n^{(l)}$, $D_n^{(l)}$, and $B_n^{(s,l)}$ are determined from the boundary conditions in the l th approximation. In the case of the first basic problem (stress moments are prescribed on the boundary L), the boundary conditions have the form

$$\begin{aligned} \left| \sigma_{\rho\rho}^{*(2k)} - i\sigma_{\rho\vartheta}^{*(2k)} \right|_L &\equiv \left| \sigma_{\rho\rho}^{(2k)} - i\sigma_{\rho\vartheta}^{(2k)} + \sigma_{\rho\rho}^{0(2k)} - i\sigma_{\rho\vartheta}^{0(2k)} \right|_L = \bar{\sigma}_{\rho\rho}^{(2k)}(\vartheta, \varepsilon) - i\bar{\sigma}_{\rho\vartheta}^{(2k)}(\vartheta, \varepsilon), \\ \sigma_{r\xi}^{*(2k-1)} \Big|_L &= \left| \sigma_{r\xi}^{(2k-1)} + \sigma_{r\xi}^{0(2k-1)} \right|_L = \bar{\sigma}_{r\xi}^{(2k-1)}(\vartheta, \varepsilon), \end{aligned} \quad (3.17)$$

where $\sigma_{\rho j'}^{0(m)}$ and $\sigma_{\rho j'}^{(m)}$ ($j' = \rho, \vartheta, \xi$) are the principal stresses and perturbations due to the presence of the hole.

Thus, in solving the boundary-value problem formulated above, it is necessary, by analogy with (3.6), to expand the principal stresses $\sigma_{\rho j'}^{0(m)}$ and the given functions $\bar{\sigma}_{r\xi}^{(2k-1)}(\vartheta, \varepsilon)$ into series in powers of the small parameter ε . In particular, if the plate is subject at infinity to constant tensile ($\sigma_{11}^{(0)\infty} = 2p_1h$, $\sigma_{22}^{(0)\infty} = 2p_2h$) and shear ($\sigma_{12}^{(0)\infty} = 2\tau h$ ($p_1, p_2, \tau = \text{const}$)) forces, then the nonzero components of the principal stress state in coordinates ρ, ϑ represented by a series in ε have the form

$$\begin{aligned} \sigma_{\rho\rho}^{0(0)} - i\sigma_{\rho\vartheta}^{0(0)} &= \frac{1}{2}(p_1h + p_2h) + \frac{1}{2}(p_1h - p_2h - 2i\tau h)e^{2i\vartheta} \\ &\times \left\{ 1 + \varepsilon[f'(\zeta) - \overline{f'(\zeta)}] - \varepsilon^2 \overline{f'(\zeta)}[f'(\zeta) - \overline{f'(\zeta)}] + \dots \right\}, \\ \sigma_{\vartheta\vartheta}^{0(0)} &= p_1h + p_2h - \sigma_{\rho\rho}^{0(0)}. \end{aligned} \quad (3.18)$$

Thus, the method outlined above allows us to determine, with required accuracy, the three-dimensional stress state of nonthin transversely isotropic plates with noncircular cavities. The solution can be refined by increasing either the number of terms in the Fourier–Legendre series in the thickness coordinate or the number of terms of the series in powers of the small parameter ε . As an illustration, let us determine the stress state of an isotropic plate with a circular cavity (zero approximation in ε) that is free from stresses ($\bar{\sigma}_{rj'}^{(m)} = 0$) and subject to uniaxial tension at infinity ($p_1 \neq 0$, $p_2 = \tau = 0$).

Table 1 summarizes, for $N = 6$ ($n = 3$), the hoop stress $\sigma_{\vartheta\vartheta}$ at the point $\rho = 1$, $\vartheta = \pi/2$ throughout the thickness of an isotropic plate with Poisson's ratio $\nu = 0.25$ for $h/R = 1$ and $h/R = 4$. For comparison, the table (lower rows) includes stresses found in [3] in solving the Kirsch problem based on the three-dimensional theory of elasticity. Table 2 shows how the transverse stress σ_{33} changes across the thickness at the same point on the cavity surface. The tables indicate that even the sixth Legendre approximation produces a solution that is in satisfactory agreement with that of the three-dimensional problem. Table 3 presents the stress components for $N = 2, 3, 4, 5, 6$. The tabulated data demonstrate the convergence of the method.

TABLE 2

h/R	σ_{33}										
	ς										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1	0.272	0.269	0.258	0.241	0.218	0.190	0.157	0.120	0.080	0.031	-0.002
	0.292	0.282	0.274	0.268	0.256	0.223	0.175	0.148	0.085	0.020	0.002
4	0.499	0.492	0.494	0.487	0.487	0.462	0.438	0.400	0.348	0.212	0.018
	0.495	0.494	0.495	0.487	0.479	0.461	0.438	0.400	0.323	0.180	0.002

TABLE 3

h/R	$\sigma_{\theta\theta}, \sigma_{33}$	ς	N					Exact
			2	3	4	5	6	
4	$\sigma_{\theta\theta}$	0	3.066	3.067	3.060	3.059	3.053	3.04
		1	2.942	2.939	2.804	2.768	2.683	2.63
	σ_{33}	0	0.469	0.516	0.454	0.495	0.499	0.495
		1	0.437	0.266	0.329	0.124	0.018	0.002

Thus, the small-parameter method can be applied to the stress-strain problem for a plate with a noncircular cylindrical cavity beginning with $N = 6$ ($n = 3$).

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