



# Solving an eigenvalue problem with a periodic domain using radial basis functions

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## ABSTRACT

A meshless method based on radial basis functions (RBFs) is proposed for solving an eigenvalue problem with a periodic domain. We compare Wendland's and Wu's compactly supported RBFs. The computational experiments examine the accuracy of the method as a result of variation in the number and layout of the original points and in addition, the shape factor. The results obtained from the method are in good agreement with the analytical solutions of the problem.

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## 1. Introduction

Often engineering problems result in a set of partial differential equations (PDEs) along with a set of boundary conditions from the process of mathematical modelling. A number of numerical methods have been developed to solve these problems including the finite-difference method (FDM), finite-volume method (FVM) and finite-element method (FEM). The FEM is a well-developed method but as it relies on elements that are connected by nodes in a predefined way it has limitations. The mesh takes time to create and often re-meshing must occur to achieve the required accuracy. Also FEM cannot model some problems well e.g., element distortion may occur under large deformations which will lead to poor accuracy. Meshless methods have been suggested as an alternative due to their capability of solving PDEs using a set of nodes within the domain of interest. One way of categorising meshless methods is according to their formulation procedures, which fall into three groups: weak-form, strong-form or collocation and a combination of weak-form and strong-form [1]. One of the earliest meshless methods in the literature is the vortex method by Chorin [2]. Other meshless methods include smooth particle hydrodynamics (SPH) [3,4], the diffuse element method (DEM) [5], the element free Galerkin (EFG) method [6], the reproducing kernel particle method (RKPM) [7], the boundary node method (BNM) [8], the meshless local Petrov–Galerkin method (MLPG) [9], the point interpolation methods (PIM) [10] and the meshfree weak-strong (MWS) form method [11].

In 1990 Kansa [12,13] introduced the radial basis function (RBF) collocation method for solving elliptical, hyperbolic and parabolic PDEs. This approach (Kansa's method) was extended to solve various PDEs including nonlinear PDEs (see, e.g., [14–16]). In 1999 Rippa [17] carried out work on selecting the correct shape parameter  $c$  for the RBFs. In 2003 Platte and Driscoll [18] showed how the global RBF collocation method could be adapted to compute eigenmodes of elliptic operators. Platte and Driscoll paid particular attention to the boundary regions and imposed Dirichlet and Neumann boundary conditions. Hardy's multiquadratics (MQ and IMQ), Duchon's Thin Plate Splines and Gaussians are the globally supported (GS) RBFs which are commonly used in the literature, for solving PDEs [19]. MQ, IMQ and Gaussian RBFs include a shape parameter, whose numerical value can be varied to control the domain of influence of the basis function. For example, in the case of the Gaussian RBF, increasing the value of the shape parameter  $c$  leads to flatter basis functions. Another class of RBF were introduced by Wendland [20], Wu [21] and Buhmann [22]. These functions are compactly supported (CS), i.e., the domain of influence extends over a finite region of the domain as opposed to the global RBFs whose influence extends over the entire domain. GSRBFs produce a dense collocation matrix  $A$ , which tends to become ill-conditioned as the number of collocation points increases. The CS kernels contain a support size parameter by which we can adjust the sparsity of the matrix, thus making  $A$  well-conditioned [23].

This paper proposes a new meshless method to solve an eigenvalue problem with periodic boundary conditions. Using CSRBFs and restricting the shape parameter to a value less than or equal to half the length of the domain ensures the periodic

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boundary conditions are satisfied. The schemes are similar to generalized finite differences but with the advantage of arbitrary point locations. Due to the radial nature of the basis functions used, the meshless method also makes no distinction regarding the dimension of the problem. In this work, numerical results are presented for several RBFs and compared to the analytical solutions for the problem.

The paper is structured as follows: Section 2 introduces RBF approximations, Section 3 presents the formulation of the meshless method and Section 4 derives the analytical solutions to the problem. The results of the computational experiments are shown in Section 5 before conclusions are drawn in Section 6.

## 2. Radial basis functions

For scattered data  $(\mathbf{x}_i, f_i) \in \mathbb{R}^{d+1}$ ,  $1 \leq i \leq N$ , the approximation  $F(\mathbf{x})$  to a function  $f(\mathbf{x})$  can be written as

$$F(\mathbf{x}) = \sum_{j=1}^N \alpha_j \phi(\|\mathbf{x} - \mathbf{x}_j\|), \quad (1)$$

where  $\|\mathbf{x} - \mathbf{x}_j\|$  is the Euclidean distance between points  $\mathbf{x}$  and  $\mathbf{x}_j$ ,  $N$  is the total number of points,  $\phi(\|\cdot\|)$  is a RBF with center  $\mathbf{x}_j$  and  $\mathbf{x}$  is a point in  $\mathbb{R}^d$ . The unknown coefficients  $\alpha_j$ ,  $j = 1, 2, \dots, N$  can be determined by setting  $F(\mathbf{x}_i) = f_i$ ,  $i = 1, 2, \dots, N$ . This gives the system of linear equations

$$\mathbf{A} \underline{\alpha} = \underline{F}, \quad (2)$$

where  $\mathbf{A} = [\phi(\|\mathbf{x}_i - \mathbf{x}_j\|)]$  is an  $N \times N$  matrix,  $\underline{\alpha} = [\alpha_j]$  and  $\underline{F} = [F(\mathbf{x}_i)]$  are  $N \times 1$  matrices. If  $\phi$  is positive definite, the matrix  $\mathbf{A}$  is non-singular, and thus the above linear system has a unique solution that can be solved, for example, by Gaussian elimination. Although this condition guarantees the uniqueness of some particular RBFs interpolants if  $\phi$  is positive definite, the RBFs generally can satisfy the condition of non-singularity. Powell [24] showed that the non-singularity of the RBFs approximations can be achieved by adding a finite number of polynomials into system (1).

Extending this to a general steady-state problem  $\mathcal{L}[u] = f$ , where  $\mathcal{L}$  is some arbitrary linear differential operator (in our case  $\mathcal{L} \equiv \nabla^2 + \lambda^2$ ), then an approximation  $\tilde{u}$  to the solution  $u$  of  $\mathcal{L}[u] = f$  can be obtained by letting

$$\tilde{u}(\mathbf{x}) = \sum_{j=1}^N \alpha_j \phi(\|\mathbf{x} - \mathbf{x}_j\|), \quad (3)$$

where  $\alpha_j$  are obtained by letting

$$\mathcal{L}[\tilde{u}](\mathbf{x}_i) = f_i, \quad 1 \leq i \leq N. \quad (4)$$

If the matrix  $\mathbf{A}_{\mathcal{L}} = [\mathcal{L}\phi(\|\mathbf{x}_i - \mathbf{x}_j\|)]$  is non-singular, then the linear system (4) has a unique solution and  $[\alpha_j]$  can be obtained by Gaussian eliminations. In general the function  $\mathcal{L}\phi(r)$  is not positive definite, even if  $\phi(r)$  is, and the theoretical proof of the solvability of (4) is still an open question. However, numerous numerical studies have shown that in many cases the matrix  $\mathbf{A}_{\mathcal{L}}$  is invertible and that  $\tilde{u}$  can provide an accurate approximation to  $u$  for sufficiently large  $N$ .

Table 1 lists some of the commonly used RBFs in the literature. These RBF's are computationally appealing and have become more and more popular in practical applications. For the compactly supported RBFs, the central idea is to use a polynomial as a function of  $r$  with a local support  $c$  (Wendland and Wu, see Table 1 for the basic definition). The existence of positive definite CSRBFs has been established by Wendland [20] and Wu [21]. Table 2 lists some of the Wendland and Wu

**Table 1**  
Radial basis functions

| Globally supported                                                      |                              |
|-------------------------------------------------------------------------|------------------------------|
| $\phi(r) = r^3$                                                         | Cubic                        |
| $\phi(r) = e^{-r^2/c}$                                                  | Gaussian                     |
| $\phi(r) = (c^2 + r^2)^{1/2}$                                           | Multiquadratic (MQ)          |
| $\phi(r) = (c^2 + r^2)^{-1/2}$                                          | Inverse multiquadratic (IMQ) |
| $\phi(r) = r^2 \log(r)$                                                 | Thin plate spline (TPS)      |
| Compactly supported                                                     |                              |
| $\phi_{\delta,\kappa}(r) = \left(1 - \frac{r}{c}\right)_+^m \cdot p(r)$ | for $\kappa \geq 1$          |

**Table 2**

Wendland and Wu CSRBFs (with  $\nabla^2 \phi \equiv (\partial^2 \phi(r)/\partial x^2) + (\partial^2 \phi(r)/\partial y^2)$ )

|                                                                                                                                                          |  |
|----------------------------------------------------------------------------------------------------------------------------------------------------------|--|
| Wendland-C2                                                                                                                                              |  |
| $\phi(r) = \left(1 - \frac{r}{c}\right)^4 \left(1 + 4\frac{r}{c}\right)$                                                                                 |  |
| $\nabla^2 \phi = -\frac{20}{c^5}(c - r)^2(2c - 5r)$                                                                                                      |  |
| Wendland-C4                                                                                                                                              |  |
| $\phi(r) = \left(1 - \frac{r}{c}\right)^6 \left(3 + 18\frac{r}{c} + 35\frac{r^2}{c^2}\right)$                                                            |  |
| $\nabla^2 \phi = -\frac{112}{c^8}(c - r)^4(c^2 + 4cr + 20r^2)$                                                                                           |  |
| Wu-C2                                                                                                                                                    |  |
| $\phi(r) = \left(1 - \frac{r}{c}\right)^5 \left(8 + 40\frac{r}{c} + 48\frac{r^2}{c^2} + 25\frac{r^3}{c^3} + 5\frac{r^4}{c^4}\right)$                     |  |
| $\nabla^2 \phi = \frac{9}{c^9}(c - r)^3(45r^4 + 135r^3c + 123r^2c^2 + 9rc^3 - 32c^4)$                                                                    |  |
| Wu-C4                                                                                                                                                    |  |
| $\phi(r) = \left(1 - \frac{r}{c}\right)^6 \left(6 + 36\frac{r}{c} + 82\frac{r^2}{c^2} + 72\frac{r^3}{c^3} + 30\frac{r^4}{c^4} + 5\frac{r^5}{c^5}\right)$ |  |
| $\nabla^2 \phi = \frac{11}{c^{11}}(c - r)^4(55r^5 + 220r^4c + 307r^3c^2 + 128r^2c^3 - 64rc^4 - 16c^5)$                                                   |  |

CSRBFs (generally used in the literature) for  $r < c$ . As usual  $r = \|\cdot\|$  denotes the Euclidean norm,  $c$  is the shape parameter,  $\delta$  is the dimension number,  $2\kappa$  is the smoothness of the function,  $(1 - r/c)_+^m$  is the truncated power function and  $p(r)$  is a prescribed polynomial.

## 3. Meshless method formulation

Consider the following PDE:

$$\nabla^2 u + \lambda^2 u = 0, \quad (5)$$

with periodic boundary conditions. In Eq. (5),  $\nabla^2$  is the Laplace operator,  $\lambda$  is a constant and the unknown function  $u$  is defined on  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  (typically  $n = 1, 2$ , or 3, when the solution to this equation makes physical sense). Eq. (5) is commonly known as the elliptic Helmholtz equation, and arises in many physical applications, in particular in acoustic and electromagnetic waves.

In two dimensions, Eq. (5) takes the form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda^2 u = 0. \quad (6)$$

Substituting Eq. (3) into (6), leads to

$$\sum_{j=1}^N \alpha_j \left( \frac{\partial^2 \phi(\|\mathbf{x} - \mathbf{x}_j\|)}{\partial x^2} + \frac{\partial^2 \phi(\|\mathbf{x} - \mathbf{x}_j\|)}{\partial y^2} \right) = -\lambda^2 \sum_{j=1}^N \alpha_j \phi(\|\mathbf{x} - \mathbf{x}_j\|), \quad i = 1, \dots, N. \quad (7)$$

Eq. (7) can be written in matrix form as

$$L\alpha = -\lambda^2 G\alpha, \quad (8)$$

where,

$$G = \begin{bmatrix} g_1(\mathbf{x}_1) & \cdots & g_N(\mathbf{x}_1) \\ \vdots & \ddots & \vdots \\ g_1(\mathbf{x}_N) & \cdots & g_N(\mathbf{x}_N) \end{bmatrix} \quad (9)$$

and

$$L = \begin{bmatrix} l_1(\mathbf{x}_1) & \cdots & l_N(\mathbf{x}_1) \\ \vdots & \ddots & \vdots \\ l_1(\mathbf{x}_N) & \cdots & l_N(\mathbf{x}_N) \end{bmatrix}, \quad (10)$$

with

$$g_j(\mathbf{x}) \equiv \phi(\|\mathbf{x} - \mathbf{x}_j\|), \quad l_j(\mathbf{x}) \equiv \nabla^2 \phi(\|\mathbf{x} - \mathbf{x}_j\|). \quad (11)$$

Eq. (8) is a generalized eigenvalue problem. The expansion coefficients  $\alpha$  are found by solving Eq. (8) (for the interior points of the domain) and (by enforcing) the periodic equations (12) and (13) at the boundary points. The domain of the system can be seen in Fig. 1.

The domain is made periodic by imposing the conditions

$$u(x, 0) = u(x, b) \quad (12)$$

and

$$u(0, y) = u(a, y), \quad (13)$$

where  $a$  and  $b$  are the length of the edges of the domain. This was calculated in the method as

$$\Delta_x = \min(|x - x_i|, a - |x - x_i|) \quad (14)$$

and

$$\Delta_y = \min(|y - y_i|, b - |y - y_i|), \quad (15)$$

where  $\Delta_x$  and  $\Delta_y$  are the difference in distances between the points in the  $x$  and  $y$  direction, respectively, and  $\min$  selects the smallest number from the two answers calculated to give the minimum distance taking into account the periodic boundary conditions. Combining Eqs. (14) and (15) the Euclidean distance was then calculated as

$$r = \sqrt{\Delta_x^2 + \Delta_y^2}. \quad (16)$$

This meshless method uses only CSRBFs as they are continuous at the boundaries of the domain (along with  $\nabla^2 \phi$ ) if the correct shape factor is selected. For a square domain the shape factor must be less than or equal to half the value of the length of the

domain to ensure that the periodic boundary conditions are satisfied.

The four CSRBFs (together with the corresponding  $\nabla^2 \phi$ ) used in the validation of the meshless method (for  $r < c$ ) are shown in Table 2.

#### 4. Analytical solutions

To validate the results of the meshless based RBFs method, analytical solutions of the elliptic equation (6) with periodic boundary conditions were obtained, by imposing the following conditions:

$$u(x, 0) = u(x, b), \quad u(0, y) = u(a, y) \quad (17)$$

and

$$u'(x, 0) = u'(x, b), \quad u'(0, y) = u'(a, y), \quad (18)$$

where  $a$  and  $b$  are the lengths of the  $x$  and  $y$  domain boundaries, respectively. Using the classical method of separation of variables, it is can be shown that the general solutions of (6) are given by

$$u(x, y) = X(x)Y(y), \quad (19)$$

where

$$X(x) = A \cos lx + B \sin lx \quad (20)$$

and

$$Y(y) = C \cos\left(y\sqrt{\lambda^2 - l^2}\right) + D \sin\left(y\sqrt{\lambda^2 - l^2}\right). \quad (21)$$

In Eqs. (20) and (21),  $l$  is some constant and

$$\lambda_{nk}^2 = \left(\frac{2\pi n}{b}\right)^2 + \left(\frac{2\pi k}{a}\right)^2, \quad n, k = 0, 1, 2, \dots \quad (22)$$

(using the boundary conditions (17) and (18)). Also when the domain is square, i.e.,  $a = b$

$$\lambda_{nk} = \frac{2\pi}{a} \sqrt{n^2 + k^2}. \quad (23)$$

#### 5. Numerical results

The computational experiments in this section were carried out using MATLAB, all results are shown for eigenvalues scaled by  $\pi$  i.e.,  $\lambda_s = \lambda/\pi$  and the domain is  $a = b = 1$ . Fig. 2 shows the layouts of the original points that are used in the generalized eigenvalue problem to calculate the expansion coefficient  $\alpha$ ; uniform, random and Sobol [25].

Fig. 3 shows how the actual relative error in the solution varies as a function of the total number of points ( $N$ ). The actual relative error  $\varepsilon_r$  is defined at the ratio of the  $L_\infty$  norm of the difference between the analytical and the numerical solutions to the  $L_\infty$  norm of the analytical solution, i.e.,

$$\varepsilon_r = \frac{\|\lambda_{\text{analytical}} - \lambda_{\text{numerical}}\|_\infty}{\|\lambda_{\text{analytical}}\|_\infty}. \quad (24)$$

This figure also shows how the layout of the original points in the domain affect the accuracy of the meshless method results. The plots are for the first eigenvalue and are compared to the first analytical answer  $\lambda_s = 2$ . It can be seen from the figure that the best results come from the uniform layout of points but that even the random layout of points still gives accurate results. The Sobol sequence adds points by placing the next point at the maximum distance from the other points in the domain. It is therefore possible to add just one extra point at a time making it more

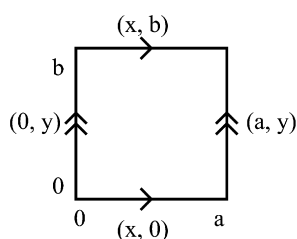


Fig. 1. Domain of system.

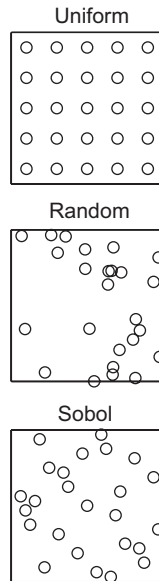
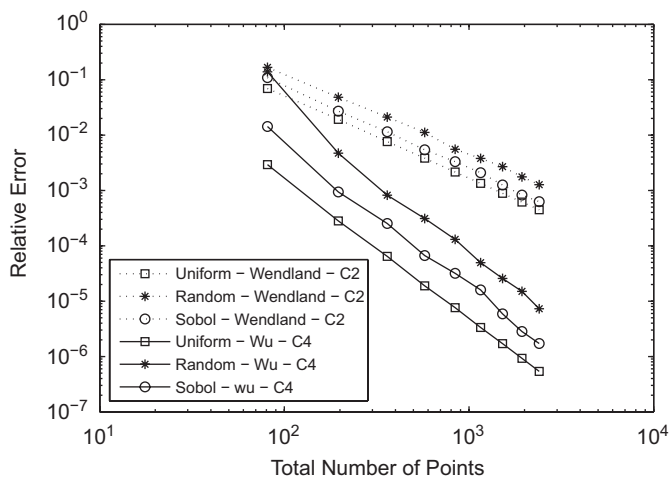


Fig. 2. Design of experiments.

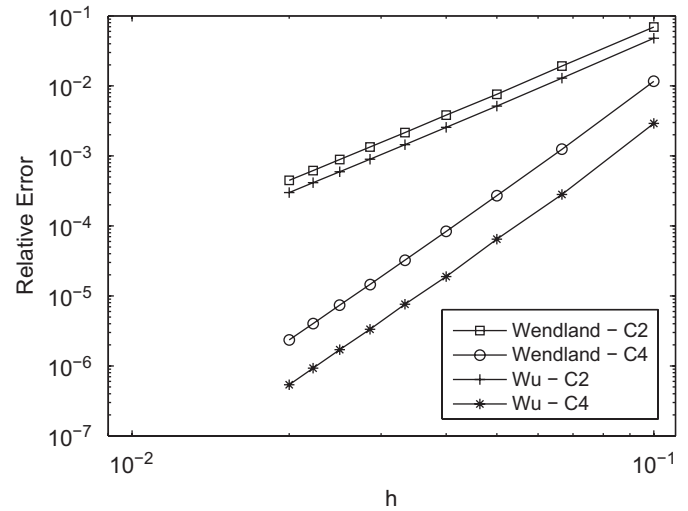
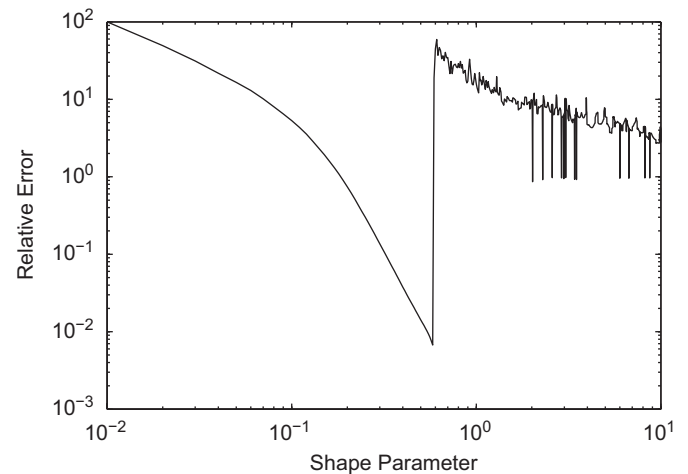
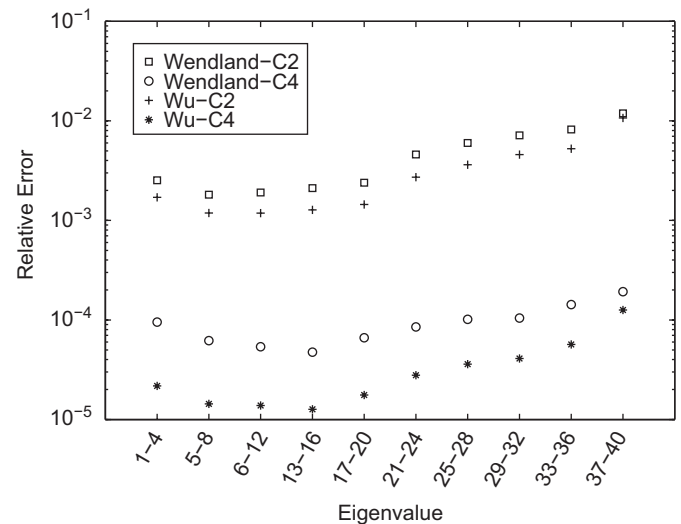
Fig. 3. Relative error between analytical answer  $\lambda_s = 2$  and meshless method result for various design of experiments.

flexible than the uniform points, which requires going up to the next grid to increase accuracy. All of the point layouts show that as the number of points is increased the accuracy of the method increases.

Fig. 4 shows the results from the experiment to obtain the convergence rate of the method, where  $h$  is the maximum mesh size. Using a straight line fit between the points, the method is found to converge at a rate of  $O(h^5)$  for the C4 functions and at a rate of  $O(h^3)$  for the C2 functions. In comparison to the central difference method, which is of order  $O(h^2)$ , the C4 functions will converge at a much faster rate than the FDM.

Fig. 5 shows the results from the experiment to find the best shape factor parameter for the method. It can be seen from this figure that the result becomes more accurate as the shape factor increases toward 0.5, which is the value of half the domain length, and highly oscillatory for  $c > 0.5$ .

Fig. 6 shows the relative errors between the analytical and meshless method results for the first forty non-zero eigenvalues which were obtained using a Sobol sequence of a 1000 points. From Fig. 6, it is worth noting that there are only 10 distinct eigenvalues (as they come in sets of four identical eigenvalues).

Fig. 4. Relative error between analytical answer  $\lambda_s = 2$  and meshless method result for various values of  $h$  using 1000 uniform points.Fig. 5. Relative error between analytical answer  $\lambda_s = 2$  and meshless method result for various values of the shape parameter using 500 uniform points.Fig. 6. Relative error between analytical answers for first 40 non-zero  $\lambda_s$  and meshless result using Sobol sequence with 1000 points.

A single point on the figure represents a set of four identical eigenvalues. From this figure it can be seen that the results obtained by Wu-C4 RBF are most accurate and the Wendland-C2 the least accurate. In addition, it can be seen that as the eigenvalue number increases the accuracy of the results decreases slightly.

## 6. Conclusion

An eigenvalue problem with periodic boundary conditions was solved in this paper using a CSRBF meshless method. The method is able to obtain accurate eigenvalues from a set of initial points in the domain of interest, using uniform, random and Sobol points. In comparison, it is found that the results obtained using uniform points are slightly better than those obtained using random or Sobol points. In addition, it is found that the Wu-C4 RBF gives the most accurate results when compared with the analytical solutions to the problem and the Wendland-C2 the least accurate. The experiments show that the best value for the shape parameter of the RBF for this problem is  $c = 0.5$ .

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