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THE INTERACTION OF LINEAR RIGID INCLUSIONS AND CRACKS

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An effective general approach in solving fracture mechanics problems of the mutual influence of crack-notches in elastic areas is the method of singular integral equations [1]. On the basis of the results of [1, 2] this article gives the basic relationships and equations characterizing the stress-strain state of an elastic plane with arbitrarily oriented linear rigid inclusions and cracks.

1. The Stress-Strain State of a Plane with Straight Rigid Inclusions and Cracks. Let us consider an isotropic plane containing N straight defects (m cracks and $N-m$ rigid inclusions) with a length of $2l_n$ ($n = 1, 2, 3, \dots, N$). Let us select the $x_n O y_n$ local system of coordinates in such a manner that the origin of the coordinates coincides with the centers z_n^0 of the defects and the $O_n x_n$ axes are directed along them. The contours of the cracks are loaded by the non-self-balancing forces

$$\sigma_n^\pm - i\tau_n^\pm = p_n(t_n) \pm q_n(t_n), \quad n = 1, 2, 3, \dots, m. \quad (1)$$

On the lines of the rigid inclusions are specified the derivatives from the displacements

$$2G \frac{d}{dt_n} (u_n^\pm + iv_n^\pm) = f_n'(t_n) \pm g_n'(t_n), \quad n = m+1, m+2, \dots, N, \quad (2)$$

and at infinitely removed points of the plate stresses and rotation are absent. The indices "+" and "-" designate the limiting stresses and displacements on the line of the defect from above and from below, respectively, G is the shear modulus, and t_n is the point on the contour of the crack or inclusion.

Introducing the designations

$$\{G_n(\rho_n^*, t_n), F_n(\rho_n^*, t_n)\} = \begin{cases} \{-2q_n(t_n), -2p_n(t_n)\}, & \rho_n^* = -1 \\ \{2g_n'(t_n), 2f_n'(t_n)\}, & \rho_n^* = \kappa, \end{cases} \quad (3)$$

where $\kappa = 3-4\nu$ for plane deformation, $\kappa = (3-\nu)/(1+\nu)$ for generalized plane stressed state, and ν is Poisson's ratio, we represent the complex potentials $\Phi(z)$ and $\Psi(z)$ [1, 2] in the following manner:

$$\Phi(z) = 1/(2\pi i) \sum_{k=1}^N (\rho_k - \rho_k^*)^{-1} \int_{-l_k}^{l_k} \frac{G_k(\rho_k, t) - G_k(\rho_k^*, t)}{t - z_k} dt; \quad (4)$$

$$\begin{aligned} \Psi(z) = & -1/(2\pi i) \sum_{k=1}^N (\rho_k - \rho_k^*)^{-1} e^{-2i\alpha_k} \int_{-l_k}^{l_k} \left[\frac{\rho_k \overline{G_k(\rho_k^*, t)} - \rho_k^* \overline{G_k(\rho_k, t)}}{t - z_k} + \right. \\ & \left. + \frac{\overline{T_k} e^{i\alpha_k} [G_k(\rho_k, t) - G_k(\rho_k^*, t)]}{(t - z_k)^2} \right] dt. \end{aligned} \quad (5)$$

Here α_k is the angle of slope of the k -th defect to the Ox axis. $T_k = te^{i\alpha_k} + z_k^0$. The parameter $\rho_k^*(\rho_k)$ acquires the values of $-1(\kappa)$ for cracks and $\kappa(-1)$ for rigid inclusions. We find the unknown functions $G_k(\rho_k, t)$ from the system of integral equations

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$$\sum_{k=1}^N \int_{-l_k}^{l_k} [Q(t, \rho_k^*, \rho_k) K_{nk}(t, x; \rho_n^*, \rho_k^*) + \overline{Q(t, \rho_k^*, \rho_k)} L_{nk}(t, x) + i \frac{G_k(\rho_k^*, t)}{t - z_k} e^{-2i\alpha_{nk}}] dt = \pi F_n(\rho_n^*, x),$$

$$|x| < l_n, \quad n = 1, 2, 3, \dots, N, \quad (6)$$

in which $\alpha_{nk} = \alpha_n - \alpha_k$

$$K_{nk}(t, x; \rho_n^*, \rho_k^*) = -i e^{i\alpha_k} [\rho_n^*/(T_k - X_n) + \rho_k^* e^{-2i\alpha_n}/(\bar{T}_k - \bar{X}_n)];$$

$$L_{nk}(t, x) = -i e^{-i\alpha_k} [1/(\bar{T}_k - \bar{X}_n) - (T_k - X_n) e^{-2i\alpha_n}/(\bar{T}_k - \bar{X}_n)^2]; \quad (7)$$

$$Q(t, \rho_k^*, \rho_k) = [G_k(\rho_k, t) - G_k(\rho_k^*, t)]/(\rho_k - \rho_k^*),$$

$$X_n = x e^{i\alpha_n} + z_n^0. \quad (8)$$

The solution of system (6) must satisfy the conditions

$$\int_{-l_n}^{l_n} Q(t, \rho_n^*, \rho_n) dt = (\rho_n^* - \rho_n)^{-1} \int_{-l_n}^{l_n} G_n(\rho_n^*, t) dt = 2(\rho_n^* - \rho_n)^{-1} (X_n + iY_n) e^{i\alpha_n}, \quad (9)$$

providing uniqueness of the displacements in the circuit of the defect contours. At the same time, let us assume that the main vector of the forces in the projections of X_n and Y_n on the Ox and Oy axes are known for each defect individually and the function $g_n(t_n)$ at the ends of the notches is equal to zero.

When the distance between the centers of the defects is sufficiently large, then the solution of Eqs. (6) may be constructed by the asymptotic method presented, for example, in [1]. The unknown function $G_n(\rho_n, x)$ is sought for in the form of the series

$$G_n(\rho_n, x) = \sum_{p=0}^{\infty} G_{np}(\rho_n, x) \lambda^p, \quad (10)$$

where

$$\lambda = 2l/d; \quad l = \max \{l_n\}; \quad d = \min \{d_{nk}\} = \min \{(z_n^0 - z_k^0) e^{-i\beta_{nk}}\}.$$

For the kernel of the system represented by relationships (7) the expansions

$$\begin{pmatrix} K_{nk}(t, x; \rho_n^*, \rho_k^*) \\ L_{nk}(t, x) \end{pmatrix} = \sum_{p=0}^{\infty} \sum_{s=0}^p \begin{pmatrix} a_{nkps} \\ b_{nkps} \end{pmatrix} t^s x^{p-s} d_{nk}^{-p-1} \quad (11)$$

are true where

$$a_{nkps} = i(-1)^{p+s} C_p^s \{\rho_n^* \exp\{-i[(p+1)\beta_{nk} - (p-s)\alpha_n - (s+1)\alpha_k]\} +$$

$$+ \rho_k^* \exp\{i[(p+1)\beta_{nk} - (p-s+2)\alpha_n - (s-1)\alpha_k]\}\};$$

$$b_{nkps} = i(-1)^{p+s+1} \{(p+1) C_p^s \exp\{i[(p+3)\beta_{nk} - (p-s+2)\alpha_n -$$

$$- (s+1)\alpha_k]\} - (C_p^s + p C_{p-1}^s) \exp\{i[(p+1)\beta_{nk} - (p-s)\alpha_n - (s+1)\alpha_k]\} -$$

$$- p C_{p-1}^{s-1} \exp\{i[(p+1)\beta_{nk} - (p-s+2)\alpha_n - (s-1)\alpha_k]\}\}, \quad (12)$$

$$C_p^s = p(p-1)(p-2) \dots (p-s+1)/s!$$

$$C_p^s = 0 \text{ with } p < s, \quad C_p^0 = 1 \text{ with } p \geq 0.$$

According to the solution found of (10) and the equation

$$K_{ln}^{\pm} - iK_{ln}^{\pm} = \mp i\rho_n^* (\rho_n - \rho_n^*)^{-1} \lim_{x \rightarrow \pm l_n} \sqrt{\pi(l_n^2 - x^2)/l_n} G_n(\rho_n, x), \quad (13)$$

we determine the stress intensity factors in the vicinity of the tip of the crack or the rigid inclusion [1, 3]. Here "+" and "-" refer to the right and left tips of the defect, respectively.

Under the action of an external load, rigid inclusions may rotate by some angle ϵ_n . If an inclusion is held from rotation, then a moment M_n must be applied to it [4]. The moments or angles of rotation for rigid inclusions are found from conditions of equality to zero, respectively, of the angle of rotation of a rigid inclusion or of the main moment of forces acting on each inclusion individually [3, 4]:

$$M_n = \rho_n^* (\rho_n^* - \rho_n)^{-1} \operatorname{Re} \left\{ \int_{-l_n}^{l_n} x [G_n(\rho_n, x)/\rho_n^* + \overline{G_n(\rho_n, x)}] dx \right\} = 0. \quad (14)$$

2. Two Arbitrarily Oriented Defects of Equal Length. In the general case of biaxial tension of a plate with two defects by the forces $\sigma_y^\infty = p$ and $\sigma_x^\infty = q$ at infinity we must assume [1, 4]

$$F_n(\rho_n^*, t_n) = 0,5 p [(1 - \rho_n^*)(1 + \eta_0) + 2(1 - \eta_0) e^{-2i\alpha_n}] + 4i G \epsilon_n, \quad (15)$$

where $\eta_0 = q/p$, $n = 1, 2$; and ϵ_n is the angle of rotation of the n -th rigid inclusion ($\epsilon_n = 0$ for a crack).

Using the above method, let us determine the function $G_n(\rho_n, x)$. After a series of cumbersome calculations on the basis of Eq. (13) we obtain expressions for the stress intensity factors, which we present here with an accuracy of $O(\lambda^4)$:

$$\begin{aligned} K_{In}^{\pm} = & (p \sqrt{\pi l/4}) \{ (1 - \rho_n^*)(1 + \eta_0) + 2(1 - \eta_0) \cos 2\alpha_n + \lambda^2 \{ (1 - \rho_k^*) \times \\ & \times (1 + \eta_0) [(\rho_n^* - 1) \cos 2(\beta - \alpha_k) + (\rho_k^* - 1) \cos 2(\beta - \alpha_n) + \\ & + 2 \cos 2(2\beta - \alpha_n - \alpha_k)] + 2(1 - \eta_0) [(\rho_n^* - 1) \cos 2\beta + \\ & + \rho_k^* \cos 2(\beta - \alpha_k - \alpha_n) + 2 \cos 2(2\beta - \alpha_n) - \cos 2(\beta - \alpha_n + \alpha_k)] \} / (16 \rho_k^*) \pm \\ & \pm (-1)^k \lambda^3 \{ (1 - \rho_k^*)(1 + \eta_0) [(\rho_n^* - 2) \cos (3\beta - \alpha_n - 2\alpha_k) + \\ & + (\rho_k^* - 1) \cos 3(\beta - \alpha_n) + 3 \cos (5\beta - 3\alpha_n - 2\alpha_k)] + \\ & + 2(1 - \eta_0) [(\rho_n^* - 2) \cos (3\beta - \alpha_n) + \rho_k^* \cos (3\beta - 3\alpha_n - 2\alpha_k) + \\ & + 3 \cos (5\beta - 3\alpha_n) - \cos (3\beta - 3\alpha_n + 2\alpha_k)] \} / (32 \rho_k^*) \} - \\ & - \lambda^2 \sqrt{\pi l G} \{ \epsilon_k [(1 - \rho_n^*) \sin 2(\beta - \alpha_k) + (\rho_k^* + 1) \sin 2(\beta - \alpha_n) - \\ & - 2 \sin 2(2\beta - \alpha_n - \alpha_k)] \pm (-1)^k \lambda \epsilon_k 0,5 [(2 - \rho_n^*) \sin (3\beta - \alpha_n - 2\alpha_k) + \\ & + (\rho_k^* + 1) \sin 3(\beta - \alpha_n) - 3 \sin (5\beta - 3\alpha_n - 2\alpha_k)] \} / (8 \rho_k^*) + O(\lambda^4); \end{aligned} \quad (16)$$

$$\begin{aligned} K_{IIIn} = & (p \sqrt{\pi l/4}) \{ 2(1 - \eta_0) \sin 2\alpha_n - \lambda^2 \{ (1 - \rho_k^*)(1 + \eta_0) [-(\rho_n^* + 1) \times \\ & \times \sin 2(\beta - \alpha_k) + (\rho_k^* - 1) \sin 2(\beta - \alpha_n) + 2 \sin 2(2\beta - \alpha_n - \alpha_k)] + \\ & + 2(1 - \eta_0) [- (1 + \rho_n^*) \sin 2\beta + \rho_k^* \sin 2(\beta - \alpha_n - \alpha_k) + \\ & + 2 \sin 2(2\beta - \alpha_n) - \sin 2(\beta - \alpha_n + \alpha_k)] \} / (16 \rho_k^*) \mp (-1)^k \lambda^3 \times \\ & \times \{ (1 - \rho_k^*)(1 + \eta_0) [-(\rho_n^* + 2) \sin (3\beta - \alpha_n - 2\alpha_k) + (\rho_k^* - 1) \times \\ & \times \sin 3(\beta - \alpha_n) + 3 \sin (5\beta - 3\alpha_n - 2\alpha_k)] + 2(1 - \eta_0) \times \\ & \times [-(\rho_n^* + 2) \sin (3\beta - \alpha_n) + \rho_k^* \sin (3\beta - 3\alpha_n - 2\alpha_k) + \\ & + 3 \sin (5\beta - 3\alpha_n) - \sin (3\beta - 3\alpha_n + 2\alpha_k)] \} / (32 \rho_k^*) \} - \\ & - 0,5 G \sqrt{\pi l} \{ 4\epsilon_n + \epsilon_k \lambda^2 [(\rho_n^* + 1) \cos 2(\beta - \alpha_k) + (\rho_k^* + 1) \cos 2(\beta - \alpha_n) - \\ & - 2 \cos 2(2\beta - \alpha_n - \alpha_k)] / (4 \rho_k^*) \pm (-1)^k \epsilon_k \lambda^3 [(2 + \rho_n^*) \cos (3\beta - \alpha_n - 2\alpha_k) + \\ & + (\rho_k^* + 1) \cos 3(\beta - \alpha_n) - 3 \cos (5\beta - 3\alpha_n - 2\alpha_k)] / (8 \rho_k^*) \} + O(\lambda^4). \end{aligned} \quad (17)$$

To calculate the moments holding rigid inclusions from rotations, on the basis of Eq. (14) we obtain the expression

$$M_n = \pi l^2 (\rho_n^* + 1) p \{ (\eta_0 - 1) \sin 2\alpha_n + \lambda^2 \{ (1 - \rho_k^*)(1 + \eta_0) [-(\rho_n^* + 1) \times$$

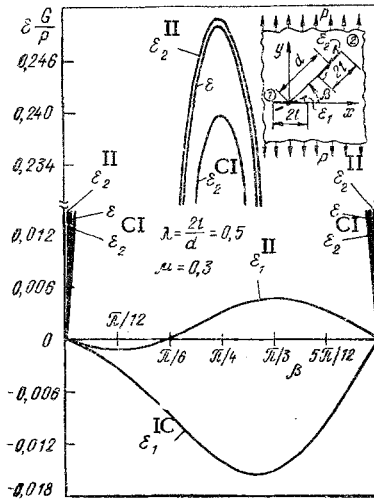


Fig. 1

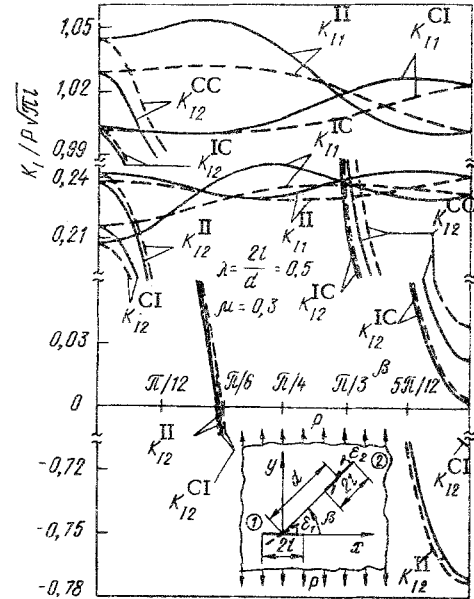


Fig. 2

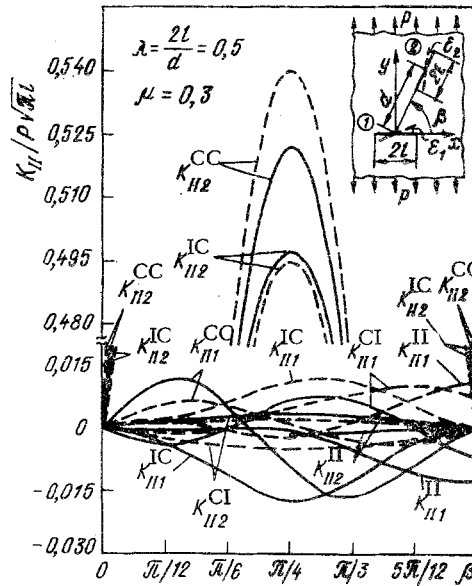


Fig. 3

$$\begin{aligned}
 & \times \sin 2(\beta - \alpha_k) + (\rho_k^* - 1) \sin 2(\beta - \alpha_n) + 2 \sin 2(2\beta - \alpha_n - \alpha_k) \Big] + \\
 & + 2(1 - \eta_0) [-(\rho_n^* + 1) \sin 2\beta + \rho_k^* \sin 2(\beta - \alpha_n - \alpha_k) + 2 \sin 2(2\beta - \alpha_n) - \\
 & - \sin 2(\beta - \alpha_n + \alpha_k)] \Big\} / (32\rho_k^*) \Big\} / (4\rho_n^*) + \pi l^2 (\rho_n^* + 1) G \{ 4\epsilon_n + \lambda^2 \epsilon_k \times \\
 & \times [(\rho_n^* + 1) \cos 2(\beta - \alpha_k) + (\rho_k^* + 1) \cos 2(\beta - \alpha_n) - \\
 & - 2 \cos 2(2\beta - \alpha_n - \alpha_k)] \Big\} / (4\rho_k^*) \Big\} / (4\rho_n^*) + O(\lambda^4). \quad (18)
 \end{aligned}$$

The angles of rotation of the inclusions ϵ_n are easy to calculate from Eq. (18), equating the moment M_n to zero.

Equations (16)-(18) reflect a series of particular cases of the interaction of pairs of different types of defects in the general case of plane tension-compression. For example, from the equations and relationships given above follow the results obtained earlier [1].

3. The Interaction of Crack-Inclusion Type Defects. From solutions (16)-(18) and taking into consideration terms of the order of λ^4 the angles of rotations and the stress intensity factors were calculated for different orientations of the defects. The cases of inter-

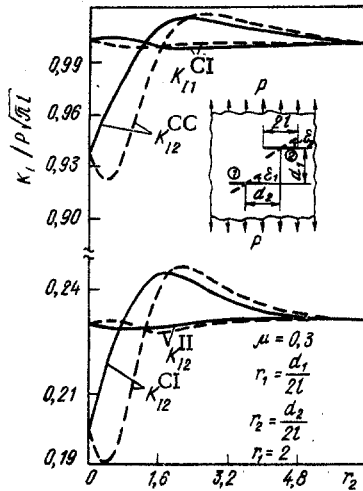


Fig. 4

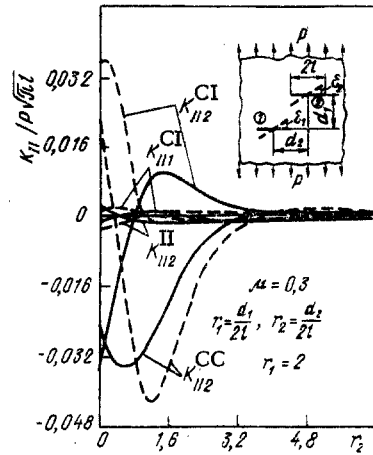


Fig. 5

action of crack-crack, crack-inclusion, and inclusion-inclusion pairs were considered individually. In the calculations $\mu = 0.3$ under conditions of the plane stressed state was accepted.

Figures 1-3 show the results of calculations with a constant $\lambda = 0.5$. The letters "C" and "I" in the superscripts with the values designate the cases where the defect is a crack or an inclusion, respectively.

Figure 1 shows graphically the change in the angles of rotations of rigid inclusions ϵ_1 and ϵ_2 in relation to the angle β in uniaxial tension of a plate with the forces $\sigma_y^\infty = p$. Here is also presented the relationship of the angle of rotation ϵ for a single rigid inclusion oriented at an angle of β to the Ox axis.

A rigid inclusion located perpendicular to the tensile forces $\sigma_y^\infty = p$ in the presence of a second defect rotates, but a crack has a less strong influence on the angle of rotation than a rigid inclusion: $|\epsilon_1^{IC}| > |\epsilon_2^{II}|$.

From Fig. 1 it may be seen that the angle of rotation of a rigid inclusion ϵ_2^{II} depends substantially upon its orientation and in the case of two inclusions it differs little from the angle ϵ for an isolated inclusion of similar orientation. A horizontally lying crack has more influence on the angle of rotation of the second defect than an inclusion since $|\epsilon_2^{CI}| > |\epsilon_2^{II}|$.

Figure 2 illustrates the change in stress intensity factors $K_{In}^\pm$ ($n = 1, 2$) in relation to the angle β . Here the second subscript designates the number of the defect. The broken lines refer to its left tip and the solid to its right.

The most unfavorable for the crack-crack pair is the orientation $\beta = 5\pi/36$. The presence of a rigid inclusion in a crack-inclusion pair with $\beta = \pi/3$ is more dangerous than the case of two interacting cracks since $K_{I1}^{CI} > K_{I1}^{CC}$. With an orientation of $\beta = \pi/6$, $K_{I2}^{II} = 0$ and $K_{I2}^{CI} = 0$, which is an indication of the minimum influence of the first defect on the intensity of stresses around the second since for an isolated rigid inclusion $K_I = 0$ with $\beta = \pi/6$.

Figure 3 provides a concept of how the stress intensity factor K_{IIIn} changes around a defect in the plans considered. Most unfavorable are the crack-crack and inclusion-crack pairs with an orientation of the second defect of $\beta = \pi/4$ since in this case K_{IIIn} reaches maximum values.

The change in the stress intensity factors K_I and K_{II} in the case of parallel defects is shown graphically in Figs. 4 and 5. In the calculations the distance between the centers of the defects $r_2 = d_2/2l$ was changed and the vertical distance between the lines of their location $r_1 = d_1/2l = 2$ was fixed.

The inclusion in the crack-inclusion and inclusion-inclusion pairs has little influence on the intensity factors K_{I1}^{CI} and K_{I2}^{II} while the presence of a crack leads to substantial deviations of the values of K_{I2}^{CI} and K_{I2}^{CC} from the corresponding stress intensity factors for isolated defects.

The development of fracture may start at the inner or the outer tips of defects depending upon their nature and mutual location.

The deviation of K_{IIIn} (Fig. 5) from the zero value ($K_{II} = 0$ in the case of an isolated crack or rigid inclusion) does not exceed 4-5%, which is an indication of the insignificant mutual influence of defects in the concrete case considered.

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AXISYMMETRIC DEFORMATION OF A SPACE WITH TWO THIN ELASTIC INCLUSIONS

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The problem of determining the field of stresses in a space containing two ellipsoidal inhomogeneities has been considered in [1, 2], but it should be noted that in both cases the solution given was very laborious.

We have obtained a comparatively simple approximate solution of this problem which agrees well with known results.

Let an infinite isotropic space contain two coaxial inclusions rigidly bound to it and bound by the oblate surfaces of rotation $V^{(1)}$ and $V^{(2)}$ symmetric relative to the center areas $S^{(1)}$ and $S^{(2)}$. The Oz axis of the cylindrical system of coordinates r, φ, z coincides with the common axis of the inclusions and the $z = 0$ plane with the plane of symmetry of the upper inclusion. It is assumed that the inclusions are different in their geometric and elastic parameters. The space is under the action of the axisymmetric tensile forces $F(r)$ which cause in the case of a uniform body in the areas $S^{(1)}$ and $S^{(2)}$, respectively, the normal $f^{(1)}(r)$ and $f^{(2)}(r)$ and tangential $g^{(1)}(r)$ and $g^{(2)}(r)$ stresses. Let us determine the stress distribution in the space in the vicinity of the inclusions.

The assumption of the thinness of the inclusions and also the results of [3, 4] make it possible to obtain for determination of the stresses in the small vicinity of their equatorial cross sections the relationships

$$\sigma_{zz}^{(i)}(r) = f^{(i)}(r) + \frac{2K_I^{(i)}}{\sqrt{\pi(\rho^{(i)} + 2r)}}, \quad \sigma_{rz}^{(i)}(r) = g^{(i)}(r) + \frac{K_{II}}{\sqrt{\pi(\rho^{(i)} + 2r)}}, \quad (1)$$

$(i = 1, 2)$

where the superscripts refer the values to the upper ("1") and lower ("2") inclusions, respectively; $\rho^{(i)}$ are the radii of the meridial cross sections of the surfaces $V^{(i)}$; $K_I^{(i)}$ and $K_{II}^{(i)}$ are the stress intensity factors for the cracks $S^{(i)}$, the edges of which are subjected to normal $f_*^{(i)}(r) - f^{(i)}(r)$ and tangential $g_*^{(i)}(r) - g^{(i)}(r)$ stresses; $f_*^{(i)}$ and $g_*^{(i)}$ are the stresses occurring on the surfaces $V^{(i)}$ under the action of the forces $F(r)$, which as a result of the thinness of the inclusions may be presented in the form

$$f^{(i)}(r) = \frac{\Delta u_z^{*(i)}(r)}{2h^{(i)}(r)} E^{(i)}, \quad g^{(i)}(r) = \frac{\Delta u_r^{*(i)}(r)}{2h^{(i)}(r)} G^{(i)}, \quad (r \in S^{(i)});$$

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