



ELSEVIER

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Theoretical and Applied Fracture Mechanics 39 (2003) 275–280

theoretical and
applied fracture
mechanics

www.elsevier.com/locate/tafmec

Stability of an infinite solid with a circular cylindrical crack under compression using the Treloar potential

A.N. Guz ^{*}, Yu.I. Khoma

*Department of Dynamics and Stability of Continuum Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine,
Nesterov Str. 3, Kyiv 03057, Ukraine*

Abstract

The fracture stability of a circular cylindrical crack in an infinite incompressible solid subjected to an axial compression is considered. A state of subcritical initial strain is assumed. The failure criterion is based on the local stability loss. The investigation is carried out in a single form for the hyper-elastic bodies with an arbitrary type of an elastic potential. Critical loads are determined for axisymmetric forms of a stability loss in the region local to the crack. The linearized problem reduced to the eigenvalue problem is solved numerically. Numerical results are obtained for solids with Treloar potential.

© 2003 Published by Elsevier Science Ltd.

1. Introduction

The problem concerning the phenomena of a subsurface convexity and delamination are sufficiently typical in geomechanics, mechanics of composites, engineering sciences. These phenomena appear, in particular, in the case of compression of cracked solids along the defects. However, it should be noted, that the problems of fracture mechanics of materials under compression along the cracks represent a special class of problems. When forces are applied along the crack, the stress intensity factors calculated on the basis of the linear fracture mechanics are equal to zero [1]. Consequently, the classical fracture criteria of the Griffith–Irwin type, the crack propagation crite-

rion [2,3] and also their generalizations used in the linear fracture mechanics are not applicable for this type of loading.

It was shown as early as in fundamental studies carried out by Euler on the critical force for compressed bars that the loss of stability, i.e. when the load reaches the critical force can be regarded as the beginning of failure. Therefore, in loading along the cracks it is rational to use, the stability criterion within the framework of the linearized stability theory proposed in [4]. Here, the fracture process is assumed to correspond with the local stability loss near the crack and quantified in terms of a critical eigenvalue.

Planar and space problems of material fracture studied on the basis of linearized formulation solids in compression along a single crack or crack arrays distributed in the same plane have been analyzed in [5–8]. The compression of materials along surface cracks has been examined in [9–12].

^{*} Corresponding author.

E-mail address: guz@carrier.kiev.ua (A.N. Guz).

The compression of a periodic system of parallel cracks and two internal parallel cracks were studied in [13].

Some failure problems have been considered in [14–19]. This paper is devoted to an axisymmetrical problem on uniform axial compression of Neo-Hookean material containing a finite cylindrical cut.

2. Basic equations

In the linearized problem, Lagrangian coordinates x_j are defined to coincide with Cartesian coordinates in the undeformed state. The components of the disturbance of the asymmetric Kirchhoff stress tensor are denoted by t_{ij} and are referred to the unit area of the undeformed body. The components of the disturbance of the displacement vector are given by u_j .

The infinite solid made of an incompressible material is subjected to uniform uniaxial pressure— p in the x_3 -direction. As a result of the compression parallel to the crack axis, a homogeneous initial (undisturbed) state takes place near the crack-like defect [14,20]

$$\begin{aligned} u_j^0 &= \delta_{jm}(\lambda_j - 1)x_m, \quad \lambda_1 = \lambda_2 \neq \lambda_3 \quad (\lambda_j = \text{const}), \\ &(\lambda_3 < 1); \\ S_{11}^0 &= S_{22}^0 = 0, \quad S_{33}^0 \neq 0 \quad (S_{33}^0 = \text{const}), \end{aligned} \quad (1)$$

where $j = 1, 2, 3$. The superscript “0” refers to the initial state such that S_{ij}^0 are the components of the symmetric stress tensor in the undisturbed state. The parameters λ_j ($j = 1, 2, 3$) are the contraction ratio, while δ_{ij} is the Kronecker symbol.

In the case of the homogeneous initial state (1), the linearized equilibrium equations for incompressible solids are [8,14]

$$\mathfrak{a}_{ij\alpha\beta} \frac{\partial^2 u_\alpha}{\partial x_i \partial x_\beta} + q_{ij} \frac{\partial p}{\partial x_i} = 0, \quad q_{ij} \frac{\partial u_j}{\partial x_i} = 0, \quad (2)$$

while the stresses can be obtained from the following relations [8]:

$$t_{ij} = \mathfrak{a}_{ij\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} + q_{ij} p, \quad q_{mx} = \delta_{mx} q_m. \quad (3)$$

Here p is a perturbation of the value p_0 incoming in governing relations for incompressible solids as

a Lagrangian coefficient corresponding to the condition of incompressibility. The components of the tensor $\tilde{\mathfrak{a}}$ for the large initial deformation theory are [8]

$$\begin{aligned} \mathfrak{a}_{ij\alpha\beta} &= \lambda_j \lambda_\alpha [\delta_{ij} \delta_{\alpha\beta} A_{ij} + (1 - \delta_{ij})(\delta_{i\alpha} \delta_{j\beta} + \delta_{i\beta} \delta_{j\alpha}) \mu_{ij}] \\ &+ \delta_{i\beta} \delta_{j\alpha} S_{\beta\beta}^0. \end{aligned} \quad (4)$$

The values μ_{ij} , A_{ij} , and consequently, components of the tensor $\tilde{\mathfrak{a}}$ are determined accordingly to the type of the elastic potential [8]

$$\begin{aligned} A_{ij} &= \frac{\partial^2 W_0}{\partial \varepsilon_i^0 \partial \varepsilon_j^0} - 2p_0 \delta_{ij} \lambda_j^{-4}, \quad S_{ii}^0 = \frac{\partial W_0}{\partial \varepsilon_i^0} + \lambda_i^{-2} p_0; \\ \mu_{ij} &= \frac{1}{2(\varepsilon_i^0 - \varepsilon_j^0)} \left(\frac{\partial}{\partial \varepsilon_i^0} - \frac{\partial}{\partial \varepsilon_j^0} \right) W_0 - p_0 \lambda_i^{-2} \lambda_j^{-2}, \\ \lambda_1 \lambda_2 \lambda_3 &= 1, \quad q_i = \lambda_i^{-1}. \end{aligned} \quad (5)$$

Let the coordinate system x_m now chosen to be a cylindrical polar system (r, θ, x_3) with the axis x_3 as one of the cylindrical crack.

The general solutions of Eqs. (2) depend on the roots n_1^0 and n_2^0 of the governing characteristic equation. For an axisymmetric problem under the homogeneous initial state (1), the general solutions in the case of unequal roots $n_1^0 \neq n_2^0$ are represented by the relations [14]

$$u_r = \frac{\partial \varphi_1}{\partial r} + \frac{\partial \varphi_2}{\partial r}; \quad u_3 = m_1 \frac{\partial \varphi_1}{\partial r} + m_2 \frac{\partial \varphi_2}{\partial r}. \quad (6)$$

The representations for the components of the Kirchhoff stress tensor $\tilde{t}$ are given by [20]

$$\begin{aligned} t_{r3} &= C_{44} \left[d_1 \frac{\partial^2 \varphi_1}{\partial r \partial x_3} + d_2 \frac{\partial^2 \varphi_2}{\partial r \partial x_3} \right]; \\ t_{rr} &= C_{44} \left[-p_1 \left(l_1 \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial x_3^2} \right) \varphi_1 \right. \\ &\quad \left. - p_2 \left(l_2 \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial x_3^2} \right) \varphi_2 \right]. \end{aligned} \quad (7)$$

Here the following notations are used:

$$\begin{aligned} C_{44} &= \mathfrak{a}_{1313}, \quad m_j = q_{11} q_{33}^{-1} n_j, \quad d_j = 1 + \mathfrak{a}_{1331} \mathfrak{a}_{1313}^{-1} m_j; \\ p_j &= [q_{11} q_{33}^{-1} (\mathfrak{a}_{1133} - \mathfrak{a}_{1313}) n_j - \mathfrak{a}_{1133} m_j + \mathfrak{a}_{3113}] \mathfrak{a}_{1313}^{-1}; \\ l_j &= (\mathfrak{a}_{1111} - \mathfrak{a}_{1122}) [q_{11} q_{33}^{-1} (\mathfrak{a}_{1133} - \mathfrak{a}_{1313}) n_j \\ &\quad - \mathfrak{a}_{1133} m_j + \mathfrak{a}_{3113}]^{-1}, \end{aligned} \quad (8)$$

where the roots n_1 and n_2 of the governing characteristic equation are

$$\begin{aligned} n_{1,2} &= c \pm \sqrt{c^2 - q_{33}^2 q_{11}^{-2} \mathfrak{e}_{3113} \mathfrak{e}_{1331}^{-1}} \\ 2q_{11}^2 \mathfrak{e}_{1331} c &= q_{11}^2 \mathfrak{e}_{3333} + q_{33}^2 \mathfrak{e}_{1111} \\ &\quad - 2q_{11} q_{33} (\mathfrak{e}_{1133} + \mathfrak{e}_{1313}). \end{aligned} \quad (9)$$

3. Problem formulation

Consider a hyper-elastic infinite solid with a circular cylindrical crack of length $2a$ and radius b . Let (r, θ, x_3) be the cylindrical coordinates with the x_3 -axis, coinciding with the axis of the cylindrical crack. Then, the crack occupies the region: $\{r = b, 0 \leq \theta < 2\pi, -a \leq x_3 \leq a\}$. The infinite solid is subjected to uniform uniaxial pressure in the x_3 -direction at infinity. As a result of the compression parallel to the crack axis, a homogeneous initial stress and strain state occurs near the crack-like defect (1).

We separate the space into two regions $r < b$ and $r > b$. Denote by the superscript “1” the quantities referred to the region $r < b$ and by the superscript “2” the ones referred to the region $r > b$.

The crack surfaces are assumed to be free of stresses. With reference to the foregoing two regions we have the following conditions:

$$\begin{aligned} t_{rr}^{(1)} &= 0, \quad t_{rr}^{(2)} = 0, \quad \text{when } r = b, \quad -a \leq x_3 \leq a; \\ t_{r3}^{(1)} &= 0, \quad t_{r3}^{(2)} = 0, \quad \text{when } r = b, \quad -a \leq x_3 \leq a. \end{aligned} \quad (10)$$

At the regions boundary outside the crack ($r = b, |x_3| > a$), the conditions of the stresses and displacements continuity should be satisfied

$$\begin{aligned} u_r^{(1)} &= u_r^{(2)}, \quad u_3^{(1)} = u_3^{(2)}, \quad \text{when } r = b, \quad |x_3| > a; \\ t_{rr}^{(1)} &= t_{rr}^{(2)}, \quad t_{r3}^{(1)} = t_{r3}^{(2)}, \quad \text{when } r = b, \quad |x_3| > a. \end{aligned} \quad (11)$$

Due to the symmetry with respect to the plane $x_3 = 0$ the boundary conditions (10) and (11) may be rewritten as follows:

$$\begin{aligned} t_{rr}^{(1)} &= t_{rr}^{(2)}, \quad t_{r3}^{(1)} = t_{r3}^{(2)}, \\ &\quad \text{when } \rho = 1, \quad 0 \leq |\varsigma| < \infty; \\ t_{rr}^{(1)} &= t_{rr}^{(2)} = 0, \quad t_{r3}^{(1)} = t_{r3}^{(2)} = 0, \\ &\quad \text{when } \rho = 1, \quad |\varsigma| < \beta; \\ u_r^{(1)} - u_r^{(2)} &= 0, \quad u_3^{(1)} - u_3^{(2)} = 0, \\ &\quad \text{when } \rho = 1, \quad |\varsigma| > \beta, \end{aligned} \quad (12)$$

where

$$\rho = r/b; \quad \varsigma = x_3/b; \quad \beta = a/b \quad (13)$$

are the dimensionless values.

4. Derivation of dual integral equations

To obtain a system of dual integral equations, the theory of the Fourier integral transforms is used.

The potential functions $\varphi_1(r, x_3)$ and $\varphi_2(r, x_3)$ referring to the regions “1” and “2” may be expressed in the form of Fourier integral transforms satisfying the regularity conditions for the displacements and stresses as $r \rightarrow \infty$.

Referred to the region “1”, there results

$$C_{44} \varphi_i^{(1)} = \int_0^\infty f_i(\lambda) \cdot I_0(\sqrt{n_i} \lambda \rho) \cdot \cos \varsigma \lambda \, d\lambda \quad (i = 1, 2). \quad (14)$$

The following applies to the region “2”:

$$C_{44} \varphi_i^{(2)} = \int_0^\infty g_i(\lambda) \cdot K_0(\sqrt{n_i} \lambda \rho) \cdot \cos \varsigma \lambda \, d\lambda \quad (i = 1, 2), \quad (15)$$

where $f_i(\lambda), g_i(\lambda)$ ($i = 1, 2$) are unknown functions; I_0 and K_0 are the modified Bessel functions. Substituting Eq. (14) into (6) and (7), it is found that

$$\begin{aligned} C_{44} b u_r^{(1)} &= \sum_{i=1}^2 \int_0^{+\infty} f_i(\lambda) \sqrt{n_i} \lambda I_1(\sqrt{n_i} \lambda \rho) \cos \varsigma \lambda \, d\lambda, \\ C_{44} b u_3^{(1)} &= - \sum_{i=1}^2 \int_0^{+\infty} f_i(\lambda) m_i \lambda I_0(\sqrt{n_i} \lambda \rho) \sin \varsigma \lambda \, d\lambda, \\ b^2 t_{r3}^{(1)} &= - \sum_{i=1}^2 \int_0^{+\infty} f_i(\lambda) d_i \sqrt{n_i} \lambda^2 I_1(\sqrt{n_i} \lambda \rho) \sin \varsigma \lambda \, d\lambda, \\ b^2 t_{rr}^{(1)} &= \sum_{i=1}^2 \int_0^{+\infty} f_i(\lambda) p_i \sqrt{n_i} \lambda^2 \left[\frac{\lambda}{\sqrt{n_i}} I_0(\sqrt{n_i} \lambda \rho) \right. \\ &\quad \left. - \frac{l_i}{\rho} I_1(\sqrt{n_i} \lambda \rho) \right] \cos \varsigma \lambda \, d\lambda. \end{aligned} \quad (16)$$

Similarly, for $r > b$ the following displacements and stresses are

$$\begin{aligned} C_{44}bu_r^{(2)} &= -\sum_{i=1}^2 \int_0^{+\infty} g_i(\lambda) \sqrt{n_i} \lambda K_1(\sqrt{n_i} \lambda \rho) \cos \varsigma \lambda d\lambda, \\ C_{44}bu_3^{(2)} &= -\sum_{i=1}^2 \int_0^{+\infty} g_i(\lambda) m_i \lambda K_0(\sqrt{n_i} \lambda \rho) \sin \varsigma \lambda d\lambda, \\ b^2 t_{r3}^{(2)} &= \sum_{i=1}^2 \int_0^{+\infty} g_i(\lambda) d_i \sqrt{n_i} \lambda^2 K_1(\sqrt{n_i} \lambda \rho) \sin \varsigma \lambda d\lambda, \\ b^2 t_{rr}^{(2)} &= \sum_{i=1}^2 \int_0^{+\infty} g_i(\lambda) p_i \sqrt{n_i} \lambda^2 \left[\frac{\lambda}{\sqrt{n_i}} K_0(\sqrt{n_i} \lambda \rho) \right. \\ &\quad \left. + \frac{l_i}{\rho} K_1(\sqrt{n_i} \lambda \rho) \right] \cos \varsigma \lambda d\lambda. \end{aligned} \quad (17)$$

Taking into account Eqs. (16) and (17), from the boundary conditions (12), we determine the functions $g_i(\lambda)$ in the terms of $f_i(\lambda)$ ($i = 1, 2$). Without going into details the rest boundary conditions yields the following system of the dual integral equations with respect to the new unknown functions $B_r(\lambda)$ and $B_3(\lambda)$

$$\begin{aligned} \int_0^{+\infty} \lambda [\alpha_{11} \cdot B_r(\lambda) + \overline{\psi_{11}}(\lambda) \cdot B_r(\lambda) + \overline{\psi_{12}}(\lambda) \cdot B_3(\lambda)] \\ \times \cos \varsigma \lambda d\lambda = 0 \quad (0 \leq \varsigma < \beta); \\ \int_0^{+\infty} \lambda [\alpha_{22} \cdot B_3(\lambda) + \overline{\psi_{21}}(\lambda) \cdot B_r(\lambda) + \overline{\psi_{22}}(\lambda) \cdot B_3(\lambda)] \\ \times \sin \varsigma \lambda d\lambda = 0 \quad (0 \leq \varsigma < \beta), \end{aligned} \quad (18)$$

$$\begin{aligned} \int_0^{+\infty} B_r(\lambda) \cos \varsigma \lambda d\lambda = 0 \quad (\beta < \varsigma < \infty); \\ \int_0^{+\infty} B_3(\lambda) \sin \varsigma \lambda d\lambda = 0 \quad (\beta < \varsigma < \infty), \end{aligned} \quad (19)$$

where α_{11} and α_{22} are the following constants:

$$\begin{aligned} \alpha_{11} &= \frac{1}{2} \cdot \frac{\sqrt{n_1} d_1 p_2 - \sqrt{n_2} d_2 p_1}{(d_1 - d_2) \sqrt{n_1} \sqrt{n_2}}, \\ \alpha_{22} &= \frac{1}{2} \cdot \frac{\sqrt{n_1} d_1 p_2 - \sqrt{n_2} d_2 p_1}{m_1 p_2 - m_2 p_1} \end{aligned} \quad (20)$$

and

$$\begin{aligned} \overline{\psi_{11}}(\lambda) = \overline{\psi_{22}}(\lambda) &= O(1/\lambda^2), \\ \overline{\psi_{12}}(\lambda) = \overline{\psi_{21}}(\lambda) &= O(1/\lambda). \end{aligned} \quad (21)$$

The obtained dual integral equations are reduced to a system of linear algebraic equations by use of the series expansion method.

5. Solution of the system of dual integral equations

To solve Eqs. (18) and (19), the unknown functions $B_r(\lambda)$ and $B_3(\lambda)$ are chosen in such manner that Eqs. (19) are automatically satisfied. Therefore, assume $B_r(\lambda)$ and $B_3(\lambda)$ to be the following infinite series with respect to the Bessel functions of the first kind J_i and the unknown constants a_j and b_j

$$\begin{aligned} B_r(\lambda) &= \sum_{j=0}^{\infty} a_j \lambda^{-1} J_{2j+1}(\beta \lambda), \\ B_3(\lambda) &= \sum_{j=0}^{\infty} b_j \lambda^{-1} J_{2j+2}(\beta \lambda). \end{aligned} \quad (22)$$

Substituting Eqs. (22) into (19), we find that the both integral equations are automatically satisfied. Substituting Eqs. (22) into (18), the following relations for $0 < \varsigma < \beta$ are found:

$$\begin{aligned} \int_0^{+\infty} \left\{ \sum_{j=0}^{+\infty} a_j [\alpha_{11} + \overline{\psi_{11}}(\lambda)] J_{2j+1}(\beta \lambda) \right. \\ \left. + \sum_{j=0}^{+\infty} b_j \overline{\psi_{12}}(\lambda) J_{2j+2}(\beta \lambda) \right\} \cos \varsigma \lambda d\lambda = 0, \\ \int_0^{+\infty} \left\{ \sum_{j=0}^{+\infty} a_j \overline{\psi_{21}}(\lambda) J_{2j+1}(\beta \lambda) \right. \\ \left. + \sum_{j=0}^{+\infty} b_j [\alpha_{22} + \overline{\psi_{22}}(\lambda)] J_{2j+2}(\beta \lambda) \right\} \sin \varsigma \lambda d\lambda = 0. \end{aligned} \quad (23)$$

The functions $\cos \varsigma \lambda$ and $\sin \varsigma \lambda$ may be expressed in the form of series as follows:

$$\begin{aligned} \cos \varsigma \lambda &= \sum_{i=0}^{\infty} \varepsilon_i J_{2i}(\beta \lambda) \cos 2i\varphi; \\ \varepsilon_0 &= 1, \quad \varepsilon_i = 2 \quad (i \geq 1); \\ \sin \varsigma \lambda &= \sum_{i=0}^{\infty} 2J_{2i+1}(\beta \lambda) \sin(2i+1)\varphi; \\ \varphi &= \arcsin(\varsigma/\beta). \end{aligned} \quad (24)$$

Substituting Eqs. (24) into (23) and putting the coefficients of $\cos 2i\varphi$ and $\sin(2i+1)\varphi$ equal to zero, the following homogeneous system of linear algebraic equations are obtained for the unknown a_j and b_j :

$$\sum_{j=0}^{+\infty} a_j [\alpha_{11} M_{ij}^{(0)}(\beta\lambda) + M_{ij}^{(1)}(\beta\lambda)] + \sum_{j=0}^{+\infty} b_j N_{ij}^{(2)}(\beta\lambda) = 0,$$

$$i = 0, 1, 2, \dots;$$

$$\sum_{j=0}^{+\infty} a_j M_{ij}^{(2)}(\beta\lambda) + \sum_{j=0}^{+\infty} b_j [\alpha_{22} N_{ij}^{(0)}(\beta\lambda) + N_{ij}^{(1)}(\beta\lambda)] = 0,$$

$$i = 0, 1, 2, \dots,$$
(25)

where the values $M_{ij}^{(k)}(\beta\lambda)$ and $N_{ij}^{(k)}(\beta\lambda)$ ($k = 0, 1, 2$) denote the following expression:

$$M_{ij}^{(1)}(\beta\lambda) = \int_0^{+\infty} \overline{\psi_{11}}(\lambda) J_{2i}(\beta\lambda) J_{2j+1}(\beta\lambda) d\lambda;$$

$$M_{ij}^{(2)}(\beta\lambda) = \int_0^{+\infty} \overline{\psi_{21}}(\lambda) J_{2i+1}(\beta\lambda) J_{2j+1}(\beta\lambda) d\lambda;$$

$$N_{ij}^{(1)}(\beta\lambda) = \int_0^{+\infty} \overline{\psi_{22}}(\lambda) J_{2i+1}(\beta\lambda) J_{2j+2}(\beta\lambda) d\lambda;$$

$$N_{ij}^{(2)}(\beta\lambda) = \int_0^{+\infty} \overline{\psi_{12}}(\lambda) J_{2i}(\beta\lambda) J_{2j+2}(\beta\lambda) d\lambda$$

$$(i, j = 0, 1, 2, \dots),$$
(26)

and also $M_{ij}^{(0)}(\beta\lambda)$ and $N_{ij}^{(0)}(\beta\lambda)$, which are reduced to

$$M_{ij}^{(0)}(\beta\lambda) = N_{ij}^{(0)}(\beta\lambda) = \begin{cases} (-1)^{j-i}/2\beta & \text{where } i \leq j; \\ (-1)^{i-j-1}/2\beta & \text{where } i > j. \end{cases}$$
(27)

For the existence of a non-trivial solution of the homogeneous system of algebraic equations (25), the determinant of the system should be set equal to zero.

$$\det \|d_{kl}(\beta, \lambda_3, \alpha)\| = 0 \quad (k, l = 1, 2, 3, \dots). \quad (28)$$

Therefore, the linearized problem has been reduced to an eigenvalue problem for Eqs. (25) in terms of the parameter $\lambda_3 < 1$.

6. Numerical results and conclusions

The problem reduced to the eigenvalue problem is solved numerically. In solving system (25), it is necessary to evaluate the infinite integrals given by Eqs. (26). Therefore, we separate finite and asymptotic parts of the integrals denoted by $M_{ij}^{(k)}(\beta\lambda)$ and $N_{ij}^{(k)}(\beta\lambda)$ ($k = 0, 1, 2$). The former is evaluated using the Gaussian-quadrature formulae.

Taking into account Eqs. (21), the latter can be written after rearranging in the following form:

$$F_k(m, n, y_0) = \int_{y_0}^{\infty} \frac{J_m(y) J_n(y)}{y^k} dy \quad (k = 1, 2), \quad (29)$$

where F_k have been evaluated by use of formulae [21]

$$F_1(m, n, y_0) = \frac{2 \sin \left[\frac{(m-n)\pi}{2} \right]}{\pi(m^2 - n^2)} + \frac{y_0}{m^2 - n^2} \times \{ J_{m+1}(y_0) J_n(y_0) - J_m(y_0) J_{n+1}(y_0) \} - \frac{J_m(y_0) J_n(y_0)}{m+n}, \quad \text{where } m \neq n;$$

$$F_1(m, m, y_0) = \frac{1}{2m} \left\{ J_0^2(y_0) + \sum_{s=1}^{m-1} J_s^2(y_0) + J_m^2(y_0) \right\},$$

$$\text{where } m = n > 0;$$

$$F_1(0, 0, y_0) = \frac{1}{\pi} \left\{ \frac{1}{y_0} + \frac{1}{192y_0^3} + \frac{121 \sin 2y_0}{96y_0} - \frac{25 \cos 2y_0}{192y_0^2} - \frac{\sin 2y_0}{192y_0^3} - \frac{121 Ci(2y_0)}{48} \right\},$$

where $m = n = 0$;

$$F_2(m, n, y_0) = \frac{1}{m+n-1} \left\{ F_1(m+1, n, y_0) + F_1(m, n+1, y_0) - \frac{J_m(y_0) J_n(y_0)}{y_0} \right\},$$

where $m+n-1 \neq 0$;

$$F_2(1, 0, y_0) = F_2(0, 1, y_0) = \frac{1}{2} \{ F_1(0, 0, y_0) - 2F_1(1, 1, y_0) - F_1(0, 2, y_0) \} + \frac{J_0(y_0) J_1(y_0)}{y_0},$$

where $m+n-1 = 0$.

(30)

The results for $m = n = 0$ are obtained by substituting the asymptotic expression for $J_0(y)$ with large value of y into the definition, and $Ci(2y)$ is a cosine integral.

The results obtained above are to applied to solids with Treloar potential (Neo-Hookean type body, unequal roots)

$$W^0 = 2C_{10}A_1 = 2C_{10}(\varepsilon_1^0 + \varepsilon_2^0 + \varepsilon_3^0), \quad (31)$$

where C_{10} is a constant of the body; $\varepsilon_i^0 = (\lambda_i^2 - 1)/2$ are the principal values of the Green's deformation tensor.

The Treloar potential in Eq. (31) may be rewritten in the simplified form

$$\begin{aligned} A_{i\beta} &= -2p_0\delta_{i\beta}\lambda_\beta^{-4}, \quad \mu_{ij} = p_0\lambda_i^{-2}\lambda_j^{-2}; \\ S_{\beta\beta}^0 &= 2C_{10} + \lambda_\beta^{-2}p_0, \quad \lambda_1\lambda_2\lambda_3 = 1; \quad q_1 = q_2 \neq q_3. \end{aligned} \quad (32)$$

Combining Eqs. (4), (8), (9) and (32), it follows that

$$\begin{aligned} \mathfrak{a}_{1111} &= 4C_{10}, \quad \mathfrak{a}_{1122} = \mathfrak{a}_{1133} = 0, \quad \mathfrak{a}_{1313} = 2C_{10}\lambda_3^{-3/2}; \\ \mathfrak{a}_{3113} &= \mathfrak{a}_{1331} = 2C_{10}, \quad \mathfrak{a}_{3333} = 2C_{10}(1 + \lambda_3^{-3}); \\ n_1 &= 1, \quad n_2 = \lambda_3^{-3}, \quad m_1 = \lambda_3^{3/2}, \quad m_2 = \lambda_3^{-3/2}; \\ l_1 &= 1, \quad l_2 = 2(1 + \lambda_3^{-3})^{-1}. \end{aligned} \quad (33)$$

Numerical evaluation gives the critical shortening λ_3^{cr} on the dimensionless parameter β^{-1} increases and tends asymptotically to the value $\lambda^* = 0.444$. This value is associated with the so-called surface instability under uniaxial compression.

References

- [1] M.K. Kassir, G.C. Sih, Mechanics of fracture, Three-dimensional Crack Problems, vol. 2, Noordhoff, Leyden, 1975.
- [2] G.P. Cherepanov, Mechanics of Brittle Fracture, Nauka, Moscow, 1974.
- [3] A.A. Wells, Application of fracture mechanics at and beyond general yielding, Brit. Weld. J. 10 (1963) 563–570.
- [4] A.N. Guz, On one criterion of fracture of solids in compression along the cracks. Spatial problem, Dokl. Acad. Nauk USSR 261 (1981) 42–45.
- [5] C.H. Wu, Plane-buckling of a crack in harmonic solid subjected to crack parallel compression, Trans. ASME J. Appl. Mech. 46 (1979) 597–604.
- [6] C.H. Wu, Plane strain buckling of crack in incompressible elastic solids, J. Elasticity 10 (1980) 163–177.
- [7] A.N. Guz, Fracture mechanics of composite materials in compression along macro cracks, Sov. Appl. Mech. 18 (1982) 489–494.
- [8] A.N. Guz, Mechanics of Brittle Fracture of Materials with Initial Stresses, Naukova Dumka, Kyiv, 1983.
- [9] L.M. Keer, S. Nemat-Nasser, A. Oranratchai, Surface instability and splitting in compressed brittle elastic solids containing crack arrays, Trans. ASME J. Appl. Mech. 49 (1982) 761–767.
- [10] A.N. Guz, V.M. Nazarenko, Symmetric failure of the half-space with penny-shaped cracks in compression, Theor. Appl. Fract. Mech. 3 (1985) 233–245.
- [11] V.M. Nazarenko, Plastic failure of materials in compression along subsurface macro cracks, Sov. Appl. Mech. 22 (1986) 245–250.
- [12] V.M. Nazarenko, Theory of failure of materials in compression along subsurface cracks in plane strain conditions, Sov. Appl. Mech. 22 (1986) 1192–1199.
- [13] A.N. Guz, V.M. Nazarenko, Failure of material in compression along a periodic system of parallel circular cracks, Sov. Appl. Mech. 23 (1987) 371–377.
- [14] A.N. Guz, Mechanics of Fracture of Composite Materials under Compression, Naukova Dumka, Kyiv, 1990.
- [15] A.N. Guz, Fundamentals of the Three-Dimensional Theory of Stability of Deformable Bodies, Springer-Verlag, 1999.
- [16] A.N. Guz, Constructing the three-dimensional theory of stability of deformable bodies, Int. Appl. Mech. 37 (2001) 1–37.
- [17] I.A. Guz, A.N. Guz, Stability of two different half-planes in compression along interfacial cracks analytical solution, Int. Appl. Mech. 37 (2001) 906–912.
- [18] A.N. Guz, I.Yu. Babich, V.N. Chekhov, The three-dimensional theory of stability of fibrous and laminated materials, Int. Appl. Mech. 37 (2001) 1103–1141.
- [19] V.L. Bogdanov, A.N. Guz, V.M. Nazarenko, Nonaxisymmetric compressive failure of a circular crack parallel to a surface of half space, Theor. Appl. Fract. Mech. 22 (1995) 239–247.
- [20] A.N. Guz, Yu.I. Khoma, V.M. Nazarenko, On fracture of an infinite elastic body in compression along a cylindrical defect, in: B.L. Karihaloo, Y.-W. Mai, M.I. Ripley, R.O. Ritchie (Eds.), Proceedings of IX International Conference on Fracture, Sydney, Australia, vol. 4, Pergamon, 1997, pp. 2047–2054.
- [21] H. Kasano, H. Matsumoto, I. Nakahara, Tension of an infinite body having a rigid cylindrical inclusion of finite length, Bull. JSME 24 (1981) 1521–1529.