

FRACTURE BY AXIAL COMPRESSION OF A CONTINUOUS CYLINDER WEAKENED BY A CYLINDRICAL CRACK NEAR ITS SURFACE

Yu. I. Khoma

UDC 539.3

Based on a nonclassical criterial approach associated with the mechanism for local loss of stability in the neighborhood of a defect, a method is proposed for solving problems related to the fracture of a compressible elastic cylinder with a cylindrical crack near its surface for large subcritical deformations. The theory of integral Fourier transforms and series expansions are used to reduce these problems to a system of paired integral equations and then to a uniform system of linear algebraic equations in the contraction parameter.

A general approach to the study of the initial stage of fracture of materials subjected to initial stresses directed along defects has been developed [1]. The proposed [2] criterion for the onset of fracture is based on the local loss of stability in the neighborhood of a defect. Fracture problems for infinite media with a cylindrical defect [3] and for bodies with a cylindrical cavity and a coaxial cylindrical crack [4] have been discussed previously in terms of this approach. In this paper we describe a method for studying the initial phase of fracture of a continuous cylinder with a cylindrical defect of finite length near its surface and subjected to uniform axial compression.

Let us consider a continuous cylinder with a circular transverse cross section of radius $b + h$ and a cylindrical crack of radius b and length $2a$ near the lateral surface of the cylinder, $\{r = b, 0 \leq \theta < 2\pi, -a \leq x_3 \leq a\}$, whose axis coincides with that of the cylinder. Here (r, θ, x_3) is a cylindrical coordinate system obtained from lagrangian coordinates x_j that are cartesian in the undeformed state, and h is the distance between the surface of the cylindrical crack in question and the lateral surface of the continuous cylinder. We shall assume that the length of the crack is sufficiently short compared to the length of the continuous cylinder.

We shall solve the linearized fracture mechanics problem in a unified form for compressible elastic materials with an arbitrary elastic potential for large subcritical deformations. In formulating the problem, we use the components of a perturbation in the asymmetric Kirchhoff stress tensor, t_{ij} , per unit area of the body in the undeformed state, as well as the perturbation components of the displacement (strain) vector u_j .

We shall assume that a continuous cylinder of elastic compressible material is subjected to a uniform uniaxial compression along the x_3 axis. Initial forces applied in this fashion create a uniform stressed-deformed subcritical state in the neighborhood of a defect [1]:

$$u_j^0 = \delta_{jm}(\lambda_j - 1)x_m, \quad \lambda_1 = \lambda_2 \neq \lambda_3 \quad (\lambda_j = \text{const}),$$

$$S_{11}^0 = S_{22}^0 = 0, \quad S_{33}^0 \neq 0 \quad (S_{33}^0 = \text{const}). \quad (1)$$

Here $j = 1, 2, 3$, the S_{ij}^0 are the components of the symmetric stress tensor per unit area of the body in its undeformed state, u_j^0 are the initial displacements, λ_j are the contractions along the axes, and δ_{jm} is the Kronecker symbol. A superscript "0" in Eq. (1) and in the following indicates quantities pertaining to the initial state. The corresponding perturbed quantities are indicated without any additional indices.

The components of the Green deformation tensor ε^0 in the theory of large initial deformations are related to the contractions λ_j by [1]

$$2\varepsilon_{ij}^0 = \delta_{ij}(\lambda_j^2 - 1). \quad (2)$$

The linearized equilibrium equations for compressible bodies in the case of a uniform subcritical state (1) have the form [1]

$$\omega_{im\alpha\beta} \frac{\partial^2 u_\alpha}{\partial x_i \partial x_\beta} = 0. \quad (3)$$

Here the components of the tensor ω are given by [1]

$$\omega_{ij\alpha\beta} = \lambda_\alpha \lambda_j \left[\delta_{ij} \delta_{\alpha\beta} A_{i\beta} + (1 - \delta_{ij}) (\delta_{i\alpha} \delta_{j\beta} + \delta_{i\beta} \delta_{j\alpha}) \mu_{ij} \right] + \delta_{i\beta} \delta_{j\alpha} S_{\beta\beta}^0, \quad (4)$$

where $A_{i\beta}$ and μ_{ij} are quantities determined by the choice of model for the material and for the uniform state (1), i.e., by the parameters λ_j .

For a hyperelastic isotropic body, where the elastic potential is given as a function of the principal values of the Green deformation tensor $\Phi_0(\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0)$, in the theory of large subcritical deformations S_{ij}^0 , $A_{i\beta}$, and μ_{ij} are defined as [1]

$$A_{ij} = \frac{\partial^2 \Phi_0}{\partial \varepsilon_i^0 \partial \varepsilon_j^0}; \quad S_{ij}^0 = \frac{\partial \Phi_0}{\partial \varepsilon_i^0}; \quad \mu_{ij} = \frac{1}{2(\varepsilon_i^0 - \varepsilon_j^0)} \left(\frac{\partial}{\partial \varepsilon_i^0} - \frac{\partial}{\partial \varepsilon_j^0} \right) \Phi_0. \quad (5)$$

The system of equilibrium equations (3) consists of partial differential equations with constant coefficients and, in the case of uniform subcritical deformations, its general solutions can be constructed in terms of potential functions. For unequal roots of the characteristic equation, $n_1^0 \neq n_2^0$, in axially symmetric problems these representations have the form

$$u_r = \frac{\partial \varphi_1}{\partial r} + \frac{\partial \varphi_2}{\partial r}; \quad u_3 = m_1 \frac{\partial \varphi_1}{\partial x_3} + m_2 \frac{\partial \varphi_2}{\partial x_3}, \quad (6)$$

where $\varphi_1(r, x_3)$ and $\varphi_2(r, x_3)$ are potential functions which satisfy the equations

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + n_i \frac{\partial^2}{\partial x_3^2} \right) \varphi_i = 0 \quad (i = 1, 2). \quad (7)$$

Here the roots of the characteristic equation are [2]

$$n_{1,2}^0 = c \pm \left(c^2 - \omega_{3333} \omega_{3113} \omega_{1111}^{-1} \omega_{1331}^{-1} \right)^{1/2};$$

$$2\omega_{1111} \omega_{1331} c = \omega_{1111} \omega_{3333} + \omega_{1331} \omega_{3113} - (\omega_{1133} + \omega_{1313})^2. \quad (8)$$

The perturbed components of the Kirchhoff stress tensor t at the cylindrical surface, which will be required below, are determined by equations of the form [4]

$$\frac{t_{r3}}{c_{44}} = d_1 \frac{\partial^2 \varphi_1}{\partial r \partial x_3} + d_2 \frac{\partial^2 \varphi_2}{\partial r \partial x_3};$$

$$\frac{t_{rr}}{c_{44}} = -p_1 \left(l_1 \frac{\partial}{\partial r} + \frac{\partial^2}{\partial x_3^2} \right) \varphi_1 - p_2 \left(l_2 \frac{\partial}{\partial r} + \frac{\partial^2}{\partial x_3^2} \right) \varphi_2. \quad (9)$$

Here

$$\begin{aligned}
c_{44} &= \omega_{1313}; \quad m_i = (\omega_{1111}n_i - \omega_{3113})(\omega_{1133} + \omega_{1313})^{-1}; \\
d_i &= 1 + m_i\omega_{1331}\omega_{1313}^{-1}; \quad p_i = (\omega_{1111}n_i + \omega_{1133}m_i)\omega_{1313}^{-1}; \\
l_i &= (\omega_{1111} - \omega_{1122})(\omega_{1111}n_i + \omega_{1133}m_i)^{-1} \quad (i = 1, 2).
\end{aligned} \tag{10}$$

We shall assume that the lateral surface of the continuous cylinder and the surface of the defect are free of stresses; the surfaces of the cylindrical crack are involved in no contact interactions. At the interface between the inner “1” ($r < b$) and outer “2” ($b < r < b + h$) regions outside the crack, the continuity conditions must hold for the displacements and stresses. Given this, the boundary conditions for the linearized problem can be written in the form

$$t_{rr}^{(2)} = 0, \quad t_{r3}^{(2)} = 0 \quad (\rho = \rho^*, \quad 0 \leq |\zeta| < +\infty); \tag{11}$$

$$t_{rr}^{(1)} = t_{rr}^{(2)} = 0, \quad t_{r3}^{(1)} = t_{r3}^{(2)} = 0 \quad (\rho = 1, \quad |\zeta| < \beta); \tag{12}$$

$$u_r^{(1)} = u_r^{(2)}, \quad u_3^{(1)} = u_3^{(2)} \quad (\rho = 1, \quad |\zeta| > \beta); \tag{13}$$

$$t_{rr}^{(1)} = t_{rr}^{(2)}, \quad t_{r3}^{(1)} = t_{r3}^{(2)} \quad (\rho = 1, \quad 0 \leq |\zeta| < +\infty), \tag{14}$$

where $\rho = r/b$, $\zeta = x_3/b$, $\beta = a/b$, and $\rho^* = (b + h)/b$ are dimensionless quantities, and the superscripts in parentheses correspond to regions “1” and “2.”

The boundary conditions, therefore, include conditions at the free surface (11) and at the surfaces of the cylindrical defect (12), as well as continuity conditions for the displacements (13) and stresses (14) at the interface between regions “1” ($r < b$) and “2” ($b < r < b + h$).

We write down the potential functions $\phi_i(r, x_3)$ which appear in representations (6) and (9) as integral Fourier expansions (with the appropriate superscripts) in both regions “1” and “2:”

$$\begin{aligned}
c_{44}\phi_i^{(1)} &= \int_0^{+\infty} v_i(\lambda) I_0(\sqrt{n_i}\lambda\rho) \cos \zeta \lambda \, d\lambda; \\
c_{44}\phi_i^{(2)} &= \int_0^{+\infty} w_i(\lambda) I_0(\sqrt{n_i}\lambda\rho) \cos \zeta \lambda \, d\lambda + \int_0^{+\infty} y_i(\lambda) K_0(\sqrt{n_i}\lambda\rho) \cos \zeta \lambda \, d\lambda \quad (i = 1, 2).
\end{aligned} \tag{15}$$

Here $v_i(\lambda)$, $w_i(\lambda)$, and $y_i(\lambda)$ ($i = 1, 2$) are unknown weighting functions, and I_0 and K_0 are the modified Bessel functions. Note that for $h \rightarrow \infty$, we must set $y_i(\lambda) \equiv 0$ in the integral representations (15) [3].

Using expansions (15), as well as representations (6) and (9), we obtain an expression for the components of the displacements and stresses in the corresponding regions:

$$\begin{aligned}
c_{44}bu_r^{(1)} &= \sum_{i=1}^2 \int_0^{+\infty} v_i(\lambda) I_1(\sqrt{n_i}\lambda\rho) \sqrt{n_i}\lambda \cos \zeta \lambda \, d\lambda, \\
c_{44}bu_3^{(1)} &= - \sum_{i=1}^2 \int_0^{+\infty} v_i(\lambda) I_0(\sqrt{n_i}\lambda\rho) m_i \lambda \sin \zeta \lambda \, d\lambda,
\end{aligned}$$

$$b^2 t_{r3}^{(1)} = - \sum_{i=1}^2 \int_0^{+\infty} v_i(\lambda) I_1(\sqrt{n_i} \lambda \rho) d_i \sqrt{n_i} \lambda^2 \sin \zeta \lambda d\lambda,$$

$$b^2 t_{rr}^{(1)} = \sum_{i=1}^2 \int_0^{+\infty} v_i(\lambda) \left[\frac{\lambda}{\sqrt{n_i}} I_0(\sqrt{n_i} \lambda \rho) - \frac{l_i}{\rho} I_1(\sqrt{n_i} \lambda \rho) \right] p_i \sqrt{n_i} \lambda \cos \zeta \lambda d\lambda; \quad (16)$$

and

$$c_{44} b u_r^{(2)} = \sum_{i=1}^2 \int_0^{+\infty} \left\{ w_i(\lambda) I_1(\sqrt{n_i} \lambda \rho) - y_i(\lambda) K_1(\sqrt{n_i} \lambda \rho) \right\} \sqrt{n_i} \lambda \cos \zeta \lambda d\lambda,$$

$$c_{44} b u_3^{(2)} = - \sum_{i=1}^2 \int_0^{+\infty} \left\{ w_i(\lambda) I_0(\sqrt{n_i} \lambda \rho) + y_i(\lambda) K_0(\sqrt{n_i} \lambda \rho) \right\} m_i \lambda \sin \zeta \lambda d\lambda,$$

$$b^2 t_{r3}^{(2)} = - \sum_{i=1}^2 \int_0^{+\infty} \left\{ w_i(\lambda) I_1(\sqrt{n_i} \lambda \rho) - y_i(\lambda) K_1(\sqrt{n_i} \lambda \rho) d_i \sqrt{n_i} \lambda^2 \sin \zeta \lambda d\lambda,\right.$$

$$b^2 t_{rr}^{(2)} = \sum_{i=1}^2 \int_0^{+\infty} \left\{ w_i(\lambda) \left[\frac{\lambda}{\sqrt{n_i}} I_0(\sqrt{n_i} \lambda \rho) - \frac{l_i}{\rho} I_1(\sqrt{n_i} \lambda \rho) \right] + \right.$$

$$\left. + y_i(\lambda) \left[\frac{\lambda}{\sqrt{n_i}} K_0(\sqrt{n_i} \lambda \rho) + \frac{l_i}{\rho} K_1(\sqrt{n_i} \lambda \rho) \right] \right\} p_i \sqrt{n_i} \lambda \cos \zeta \lambda d\lambda. \quad (17)$$

Substituting Eqs. (16) and (17) in the boundary conditions (11) and (14), we obtain equations that couple the unknown weighting functions. Given these relationships, the remaining boundary conditions (12) and (13) yield a system of paired integral equations

$$\int_0^{+\infty} \left\{ \psi_{11}(\lambda) D_r(\lambda) + \psi_{12}(\lambda) D_3(\lambda) \right\} \cos \zeta \lambda d\lambda = 0, \quad 0 \leq \zeta < \beta;$$

$$\int_0^{+\infty} \left\{ \psi_{21}(\lambda) D_r(\lambda) + \psi_{22}(\lambda) D_3(\lambda) \right\} \sin \zeta \lambda d\lambda = 0, \quad 0 \leq \zeta < \beta; \quad (18)$$

$$\int_0^{+\infty} D_r(\lambda) \cos \zeta \lambda d\lambda = 0, \quad \int_0^{+\infty} D_3(\lambda) \sin \zeta \lambda d\lambda = 0, \quad \zeta > \beta \quad (19)$$

in the new unknown functions $D_r(\lambda)$ and $D_3(\lambda)$.

We expand the unknown functions $D_r(\lambda)$ and $D_3(\lambda)$ in infinite series of Bessel functions of the first kind with unknown coefficients a_j and b_j [5]:

$$D_r(\lambda) = \sum_{j=0}^{+\infty} a_j \lambda^{-1} J_{2j+1}(\beta \lambda); \quad D_3(\lambda) = \sum_{j=0}^{+\infty} b_j \lambda^{-1} J_{2j+2}(\beta \lambda). \quad (20)$$

Substituting the unknown functions $D_r(\lambda)$ and $D_3(\lambda)$ in Eq. (19), we obtain some identities [6] and from Eqs. (16) we have

$$\int_0^{+\infty} \left\{ \sum_{j=0}^{+\infty} a_j \psi_{11}(\lambda) J_{2j+1}(\beta\lambda) + \sum_{j=0}^{+\infty} b_j \psi_{12}(\lambda) J_{2j+2}(\beta\lambda) \right\} \lambda^{-1} \cos \zeta \lambda d\lambda = 0;$$

$$\int_0^{+\infty} \left\{ \sum_{j=0}^{+\infty} a_j \psi_{21}(\lambda) J_{2j+1}(\beta\lambda) + \sum_{j=0}^{+\infty} b_j \psi_{22}(\lambda) J_{2j+2}(\beta\lambda) \right\} \lambda^{-1} \sin \zeta \lambda d\lambda = 0, \quad 0 \leq \zeta < \beta. \quad (21)$$

Using the series representations of the functions $\cos \zeta \lambda$ and $\sin \zeta \lambda$ [6],

$$\sin \zeta \lambda = \sum_{i=0}^{+\infty} 2 J_{2i+1}(\beta\lambda) \sin(2i+1)\varphi;$$

$$\cos \zeta \lambda = \sum_{i=0}^{+\infty} \varepsilon_i J_{2i}(\beta\lambda) \cos 2i\varphi, \quad (22)$$

where $\varepsilon_0 = 1$, $\varepsilon_i = 2$ ($i \geq 1$), and $\varphi = \arcsin(\zeta/\beta)$, and then equating the coefficients of $\cos 2i\varphi$ and $\sin(2i+1)\varphi$ in Eq. (21) to zero, we obtain a homogeneous system of linear algebraic equations in the unknowns a_j and b_j ,

$$\sum_{j=0}^{+\infty} a_j \Phi_{ij}^{11}(\beta\lambda) + \sum_{j=0}^{+\infty} b_j \Phi_{ij}^{12}(\beta\lambda) = 0;$$

$$\sum_{j=0}^{+\infty} a_j \Phi_{ij}^{21}(\beta\lambda) + \sum_{j=0}^{+\infty} b_j \Phi_{ij}^{22}(\beta\lambda) = 0 \quad (i = 0, 1, 2, \dots). \quad (23)$$

Here

$$\Phi_{ij}^{11}(\beta\lambda) = \int_0^{+\infty} \psi_{11}(\lambda) J_{2i}(\beta\lambda) J_{2j+1}(\beta\lambda) \lambda^{-1} d\lambda;$$

$$\Phi_{ij}^{12}(\beta\lambda) = \int_0^{+\infty} \psi_{12}(\lambda) J_{2i}(\beta\lambda) J_{2j+2}(\beta\lambda) \lambda^{-1} d\lambda;$$

$$\Phi_{ij}^{21}(\beta\lambda) = \int_0^{+\infty} \psi_{21}(\lambda) J_{2i+1}(\beta\lambda) J_{2j+1}(\beta\lambda) \lambda^{-1} d\lambda;$$

$$\Phi_{ij}^{22}(\beta\lambda) = \int_0^{+\infty} \psi_{22}(\lambda) J_{2i+1}(\beta\lambda) J_{2j+2}(\beta\lambda) \lambda^{-1} d\lambda. \quad (24)$$

The homogeneous system of algebraic equations (23) has a nontrivial solution if the determinant of the system equals zero, i.e.,

$$\det \|d_{kl}(\beta, \rho^*, \lambda_3, \omega)\| = 0 \quad (k, l = 1, 2, 3, \dots). \quad (25)$$

Thus, determining the critical loads reduces to solving eigenvalue problems for the contraction parameter $\lambda_3 < 1$ in the system of homogeneous algebraic equations (23), which is solved numerically.

REFERENCES

1. A. N. Guz', *Mechanics of Brittle Fracture of Materials with Initial Stresses* [in Russian], Kiev (1983).
2. A. N. Guz', *Elements of the Three-Dimensional Theory of Stability of Deformed Bodies* [in Russian], Kiev (1986).
3. A. N. Guz', V. M. Nazarenko, and Yu. I. Khoma, "Fracture of an infinite incompressible hyperelastic material during compression along a cylindrical crack," *Prikl. Mekh.*, **32**, No. 5, 3–10 (1996).
4. Yu. I. Khoma, "Method for studying fracture of bodies with a cylindrical cavity and a coaxial cylindrical crack of finite length under axial compression," *Teor. Prikl. Mekh.*, No. 30, 76–84 (1999).
5. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, NBS, Washington, DC (1979).
6. I. S. Gradshtein and I. M. Ryzhik, *Tables of Integrals, Sums, Series, and Products* [in Russian], 4th ed., Moscow (1963).